THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR THE DIRAC OPERATOR ON GLOBALLY HYPERBOLIC MANIFOLDS WITH TIMELIKE BOUNDARY

NADINE GROSSE AND SIMONE MURRO

ABSTRACT. We consider the Dirac operator on globally hyperbolic manifolds with timelike boundary and show well-posedness of the Cauchy initial boundary value problem coupled to MIT-boundary conditions. This is achieved by transforming the problem locally into a symmetric positive hyperbolic system, proving existence and uniqueness of weak solutions and then using local methods developed by Lax, Phillips and Rauch, Massey to show smoothness of the solutions. Our proof actually works for a slightly more general class of local boundary conditions.

1. INTRODUCTION

The well-posedness of the Cauchy problem for the Dirac operator on a Lorentzian manifold $(\overline{\mathcal{M}}, g)$ is a classical problem which has been exhaustively studied in many contexts. If the underlying background is *globally hyperbolic*, a complete answer is known: In [26] it was shown that a fundamental solution for the Dirac equation can obtained from a fundamental solution of the conformal wave operator via the Lichnerowicz formula. Since the Cauchy problem for the conformal wave operator is well-posed [16, 43], it follows that the Cauchy problem for the Dirac operator is also well-posed. See also [6] for a direct treatment of Cauchy problems for Dirac operators.

Even if there exists a plethora of models in physics where globally hyperbolic spacetimes have been used as a background, there also exist many applications which require a manifold with non-empty boundary. Indeed, recent developments in quantum field theory focused their attention on manifolds with timelike boundary [10, 50], e.g. anti-de Sitter spacetime [19, 20] and BTZ spacetime [13]. Moreover, experimental setups for studying the Casimir effect enclose (quantum) fields between walls, which may be mathematically described by introducing timelike boundaries [24]. Also moving walls in a spatial set-up correspond to a timelike boundary in the Lorentzian manifold. In these settings, the correspondence between the well-posedness of the conformal wave operator and the Dirac operator breaks down due to the boundary condition. Even if the Cauchy problem for the conformal wave operator is proved to be well-posed for a large class of boundary conditions for stationary spacetimes [18], it is still not clear how to relate this with the Cauchy problem for the Dirac operator. In fact, solvability in the Dirac case very much depends on the boundary condition, e.g. if the Dirichlet boundary condition is

applied to spinors on the boundary, in general there does not exist any smooth solution to the Dirac equation.

The goal of this paper is to investigate the well-posedness of the Cauchy problem for the Dirac operator in globally hyperbolic manifolds with timelike boundary $\mathcal{M}$ in the sense of Definition 2.1. Coming from the 'moving wall'-picture our time function and splitting should by induced from an exterior globally hyperbolic manifold: Let $(\overline{\mathcal{M}}, g)$ be a globally hyperbolic spin manifold of dimension $n + 1$. Let $\mathcal{N}$ be an n-dimensional Lorentzian submanifold of $\overline{\mathcal{M}}$. We assume additionally that $\mathcal{N}$ divides $\overline{\mathcal{M}}$ into two (not empty) connected components. In Section 2.1 we see that each of these connected components is then globally hyperbolic manifolds with timelike boundary and that every globally hyperbolic manifolds with timelike boundary arises that way. In particular, there we see that any Cauchy time function $t \colon \overline{\mathcal{M}} \to \mathbb{R}$ on $\overline{\mathcal{M}}$ induces one on the connected component. Thus we can work with the splitting and time function induced from $\overline{\mathcal{M}}$ in the following: Then $\overline{\Sigma}_s := t^{-1}(s)$ is a smooth spacelike Cauchy surface of $\overline{\mathcal{M}}$ and, see Section 2.1, $\widehat{\Sigma} := \overline{\Sigma} \cap \mathcal{N}$ is a Cauchy surface for $\mathcal{N}$. In particular, $\{t^{-1}(s)\}_{s \in \mathbb{R}}$ gives a foliation by Cauchy surfaces, and we set $\Sigma_s := t^{-1}(s) \cap \mathcal{M}$.

We always assume that we fix the spin structure on $\mathcal{M}$. Let $S\mathcal{M}$ denote the spinor bundle over $\mathcal{M}$ and $S\Sigma_0$ the induced spinor bundle over Σ_0. Moreover, let D be the Dirac operator on $S\mathcal{M}$, for details and notation see Section 2.2.

Our main result is a well-posedness theorem for the Dirac operator with MIT-boundary conditions. The *MIT boundary condition* is a local boundary condition that was introduced for the first time in [15] in order to reproduce the confinement of quark in a finite region of space: "Dirac waves" are indeed reflected on the boundary. Few years later, it was used in the description of hadronic states like baryons [14] and mesons [40]. More recently, the MIT boundary condition were employed in [48] for computing the Casimir energy in a a three-dimensional rectangular box, in [29] and [32, 33] in order to construct an integral representation for the Dirac propagator in Kerr-Newman and Kerr Black Hole Geometry respectively, and in [41] for proving the asymptotic completeness for linear massive Dirac fields on the Schwarzschild Anti-de Sitter spacetime.

In this paper we obtain

Theorem 1.1. *The Cauchy problem for the Dirac operator with MIT-boundary condition on a globally hyperbolic spin manifold $\mathcal{M}$ with timelike boundary $\partial\mathcal{M}$ is well-posed, i.e., for any $f \in \Gamma_{cc}(S\mathcal{M})$ and $h \in \Gamma_{cc}(S\Sigma_0)$ there exists a unique smooth solution ψ with spatially compact support to the mixed initial-boundary value problem*

$$\begin{cases} \mathsf{D}\psi = f \\ \psi|_{\Sigma_0} = h \\ (\gamma(\mathbf{n}) - \imath)\psi|_{\partial\mathcal{M}} = 0 \end{cases} \tag{1.1}$$

which depends continuously on the data (f, h). Here, $\gamma(\mathbf{n})$ denotes Clifford multiplication with $\mathbf{n}$ the outward unit normal on $\partial\mathcal{M}$ and $\Gamma_{cc}(.)$ denotes the space of sections that are compactly supported in the interior of the underlying manifold.

For a subclass of stationary spacetimes with timelike boundary admitting a suitable timelike Killing vector field Theorem 1.1 was already proven in [32].

Remark 1.2. Theorem 1.1 holds under more general local boundary conditions, namely replacing $\mathsf{M} = \gamma(\mathbf{n}) - \imath$ by any linear non-invertible map $\mathsf{M}\colon \Gamma(S\partial\mathcal{M}) \to \Gamma(S\partial\mathcal{M})$ with constant kernel dimension and such that $\mathsf{M}\psi|_{\partial\mathcal{M}} = 0$ and $\mathsf{M}^\dagger\psi|_{\partial\mathcal{M}} = 0$, see Remark 3.2 for the definition of the adjoint boundary condition $\mathsf{M}^\dagger$, both imply

$$\langle \psi \,|\, \gamma(e_0)\gamma(\mathbf{n})\psi\rangle_q = 0 \tag{1.2}$$

for all $q \in \partial\mathcal{M}$, see also Remark 3.19. Here e_0 is the globally defined unit timelike future pointing vector field defined by (2.1).

As usual, the well-posedness of the Cauchy problem will guarantee the existence of *Green operators* for the Dirac operator. In globally hyperbolic spin manifolds with empty boundary, these operators play a fundamental role in the quantization of linear field theory [22, 26]. In loc. cit., the quantization of a free field theory is interpreted as a two-step procedure: The first consists of the assignment to a physical system of a *-algebra of observables which encodes structural properties such as causality, dynamics and the canonical anticommutation relations. The second step calls for the identification of an algebraic state, which is a positive, linear and normalized functional on the algebra of observables. This quantization scheme goes under the name of *algebraic quantum field theory* and it is especially well-suited for formulating quantum theories for Green-hyperbolic operators also on manifold – see e.g. [12, 36] for textbook, to [5, 8, 28] for recent reviews, and [9, 21, 22, 23, 25, 30, 31] for some applications.

Remark 1.3. The Cauchy problem (1.1) is still well-posed in a larger class of initial data. But then some compatibility condition for f and h on $\partial\Sigma_0$ is needed—see Remark 3.15. In the subclass of stationary spacetimes considered in [32], as mentioned above, this compatibility condition reduces to the one therein. Without these compatibility conditions the solution would still exist but the singularities contained in a neighborhood of $\partial\Sigma_0$ would propagate with time along lightlike geodesics. In case, they hit again the boundary some boundary phenomena as reflection will occur. For future work it is of course interesting to obtain an explicit method to construct the corresponding Green operators and to obtain more information on how singularities behave when hitting the boundary.

Our strategy to prove the well-posedness of the Cauchy problem is as follows: Firstly we restrict to a strip of finite time and prove general properties of smooth solutions as finite propagation of speed and uniqueness in Section 3.2. Using the theory of symmetric positive hyperbolic systems, see e.g. [35], we investigate properties of a weak solution in Section 3.3. In Section 3.4 we localize the problem by introducing suitable coordinates small enough that we can associate to our Dirac problem a hyperbolic system that fits into the class considered in [42, 47] and thus provides smoothness of a weak solution. Then we can use these properties to prove existence of a weak solution in an arbitrary time strip. With the help of uniqueness we can then easily glue together solutions on an arbitrary time strip to obtain global ones in Section 4.

2. Preliminaries

For a Lorentzian manifold $\mathcal{M}$, we use the convention that the metric g of $\mathcal{M}$ has signature$(-, +, \cdots, +)$.

Globally hyperbolic as in [6, 37] means that the manifold admits a Cauchy surface, i.e. a hypersurface that is hit exactly once by every inextendable timelike curve. Moreover, this surface can always be chosen to be smooth and spacelike, see [11]. Without further mentioning it, we always assume that our manifolds are orientable and time-orientable. Note that for $n + 1 = 4$, the case relevant for physics, such globally hyperbolic manifolds are automatically spin.

2.1. Globally Hyperbolic Manifolds with Timelike Boundary.
The definition of what should be a globally hyperbolic manifold with timelike boundary was first proposed by Solis in [49], see also [17].

Definition 2.1. A *globally hyperbolic manifold with timelike boundary* is an $(n+1)$-dimensional, oriented, time-oriented Lorentzian manifold $(\mathcal{M}, g)$ with boundary $\partial \mathcal{M}$ such that

  (i) the pullback of g respect the natural inclusion $\iota \colon \partial \mathcal{M} \to \mathcal{M}$ defines a Lorentzian metric $\iota^* g$ on the boundary;
  (ii) $\mathcal{M}$ is causal, i.e. there are no closed causal curves;
  (iii) for every point $p, q \in \mathcal{M}$, $J^+(p) \cap J^-(q)$ is compact, where $J^+(p)$ (*resp.* $J^-(p)$) denotes the causal future (*resp.* past) of $p \in M$.

In [2] Aké, Flores and Sanchez proved that for a globally hyperbolic manifold with timelike boundary there always exists a Cauchy time function (i.e., a continuous function $\mathcal{M} \to \mathbb{R}$ which increases strictly on any future-directed causal curve and whose level sets are Cauchy surfaces, see [7, p. 65]) whose gradient is tangent to the boundary and obtain the analogue of the usual spacetime splitting of globally hyperbolic manifolds.

Thinking of the occurrence of manifolds with timelike boundary as described in the introduction, e.g. 'moving mirrors', it is more natural for us, to obtain our manifold as a part of a globally hyperbolic manifold without boundary and to work with the induced time function. We prove in the following that under natural assumptions this always leads to a globally hyperbolic manifold with timelike boundary as defined above and all such manifolds are obtained that way.

In particular, we will see that we obtain [2, Thm. 1.1] except for the condition that the gradient of the Cauchy time function is tangent to the boundary. Most of this can of course be found along the lines of the proofs in [17, 2, 1]. Especially that every globally hyperbolic manifold with timelike boundary arises that way is shown in proven in [2, Cor. 5.8]. But for further use we decided to still formulate the following theorems here.

Theorem 2.2. *Let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be a globally hyperbolic manifold (without boundary). Let $\iota \colon \mathcal{N} \hookrightarrow \widetilde{\mathcal{M}}$ be an embedding of a timelike hypersurface such that $\widetilde{\mathcal{M}} \setminus \mathcal{N}$ has two connected components $\mathcal{M}_i$, $i = 1, 2$. Then, $\overline{\mathcal{M}}_i = \mathcal{M}_i \cup \mathcal{N} \subset \widetilde{M}$ is a globally hyperbolic manifold with timelike boundary and $(\mathcal{N}, \widetilde{g}|_{\mathcal{N}})$ is globally hyperbolic. Moreover, each globally hyperbolic manifold with timelike boundary arises as a $\overline{\mathcal{M}}_i$ in a construction as above.*

Proof. Property (i) in Definition 2.1 is satisfied since $\overline{\partial \mathcal{M}_i} := \mathcal{N}$ is timelike by assumption. Moreover, every closed causal curve in $\mathcal{N}$ or $\overline{\mathcal{M}}_i$ is one in $\mathcal{M}$ and, hence, does not exist. In order to show that property (iii) is satisfied, we first show that $\mathcal{N}$ and $\overline{\mathcal{M}}_i$ are strongly causal. Notice that, since $(\widetilde{M}, \widetilde{g})$ is globally hyperbolic, it is in particular strongly causal [7, p. 59]. We first show, that $(\mathcal{N}, \widetilde{g}|_{\mathcal{N}})$ is strongly causal as well: Let $p \in \mathcal{N}$ and let U be an open neighborhood of p in $\mathcal{N}$. Choose an open neighborhood $\widetilde{U}$ of p in $\widetilde{\mathcal{M}}$ such that $\widetilde{U} \cap \mathcal{N} = U$. Since $\widetilde{\mathcal{M}}$ is strongly causal, there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of p in $\widetilde{\mathcal{M}}$ such that any causal curve starting and ending in $\widetilde{V}$ is completely contained in $\widetilde{U}$. Let now $V := \widetilde{V} \cap \mathcal{N}$. Then any causal curve in $\mathcal{N}$ starting and ending in V is in particular a causal curve of $\widetilde{\mathcal{M}}$ starting and ending in $\widetilde{V}$ and hence completely stays in $U = \widetilde{U} \cap \mathcal{N}$. Analogously, we see that $(\overline{\mathcal{M}}_i, \widetilde{g})$ is strongly causal.

Furthermore, we note that both $\mathcal{N}$ and $\overline{\mathcal{M}}_i$ inherit the time-orientation from $\widetilde{\mathcal{M}}$ and that $\mathcal{N}$ has to be closed since $\mathcal{M} \setminus \mathcal{N}$ has two connected components.

Let $p, q \in \mathcal{N}$. Let $J_{\pm}^{\mathcal{N}}(.)$ denote that causal past/future inside $\mathcal{N}$ and $J_{\pm}^{\widetilde{\mathcal{M}}}(.)$ the ones inside $\widetilde{\mathcal{M}}$. Note that $J_{\pm}^{\mathcal{N}}(p) \subset J_{\pm}^{\widetilde{\mathcal{M}}}(p) \cap \mathcal{N}$. Hence $J_{-}^{\mathcal{N}}(p) \cap J_{+}^{\mathcal{N}}(q)$ is a subset of the compact set $J_{-}^{\widetilde{\mathcal{M}}}(p) \cap J_{+}^{\widetilde{\mathcal{M}}}(q) \cap \mathcal{N}$. This implies that the closure of $J_{-}^{\mathcal{N}}(p) \cap J_{+}^{\mathcal{N}}(q)$ is compact in $\mathcal{N}$. Together with the strong causality of $\mathcal{N}$ [7, Lemma 4.29] implies that $J_{-}^{\mathcal{N}}(p) \cap J_{+}^{\mathcal{N}}(q)$ is already compact and, hence, $(\mathcal{N}, \widetilde{g}|_{\mathcal{N}})$ is globally hyperbolic. We note that [7, Lemma 4.29] has exactly the same proof if we allow the manifold to have boundary. Hence, the analogue arguments from above imply that $(\overline{\mathcal{M}}_i, \widetilde{g})$ is globally hyperbolic as well.

Let now $(\mathcal{M}, g)$ be a globally hyperbolic manifold with timelike boundary. As stated before this is [2, Cor. 5.8]. To provide a short-cut let us shortly present the strategy of the proof: Every Lorentzian manifold $(\mathcal{M}, g)$ with boundary admits an extension to a Lorentzian manifold $(\hat{\mathcal{M}} := \mathcal{M} \cup_{\partial \mathcal{M}} (\partial \mathcal{M} \times [0, \varepsilon)), \hat{g})$ such that $\hat{g}|_{\mathcal{M}} = g$ [49, Theorem A.1]. On a given manifold with boundary the set of metrics that make this manifold into a globally hyperbolic one with timelike boundary is open in the set of Lorentzian metrics in the 'time cone topology' (defined by the $<$-relation on the space of Lorentzian metric as in [37, Section 6], see also [2, below Def. 2.3]) and hence in the fine C^0-topology: This can be seen exactly along the lines of the proof of Geroch in [37]. Hence, there is a smooth function $\delta \colon \partial \mathcal{M} \to (0, \varepsilon)$ such that $(\mathcal{M}_1 := \mathcal{M} \cup_{\partial \mathcal{M}} \{(x, t) \mid x \in \partial \mathcal{M}, 0 < t \leq \delta(x)\}, \hat{g})$ is a globally hyperbolic manifold with timelike boundary. Now we can deform $\hat{g}$ on a small enough neighborhood of $\partial \mathcal{M}_1$ to have product structure on this subset of $\mathcal{M}_1 \setminus \mathcal{M}$ and such that the new metric g_1 is still globally hyperbolic on $\mathcal{M}_1$. The double $(\hat{\mathcal{M}}_1, \hat{g}_1)$ of $(\mathcal{M}_1, g_1)$ is then a smooth Lorentzian manifold without boundary with $(\mathcal{M}, g) \subset (\hat{\mathcal{M}}_1, \hat{g}_1)$. The double $(\hat{\mathcal{M}}_1, \hat{g}_1)$ is itself globally hyperbolic since every causal curve between two points in $\partial \mathcal{M}_1 \subset \hat{\mathcal{M}}_1$ can be mapped using the reflection symmetry of the double to a causal curve staying in $\mathcal{M}_1$. $\square$

Theorem 2.3. *Let $(\widetilde{\mathcal{M}}, \widetilde{g})$ be a globally hyperbolic manifold (without boundary). Let $\iota \colon \mathcal{N} \hookrightarrow \widetilde{\mathcal{M}}$ be the embedding of a timelike hypersurface such that $\widetilde{\mathcal{M}} \setminus \mathcal{N}$ has two connected components $\mathcal{M}_i$, $i = 1, 2$. Let $t \colon \widetilde{M} \to \mathbb{R}$ be a Cauchy time function for $\widetilde{M}$. Then, $t^{-1}(s) \cap \mathcal{N}$ is a Cauchy surface of $\mathcal{N}$ and $t^{-1}(s) \cap \overline{\mathcal{M}}_i$ one of $\overline{\mathcal{M}}_i$ for all $s \in \mathbb{R}$. Moreover, $t|_{\mathcal{N}} \colon \mathcal{N} \to \mathbb{R}$ is a Cauchy time time function of $\mathcal{N}$.*

Proof. From Theorem 2.2 we know that $\mathcal{N}$ with its induces metric is globally hyperbolic.

For any s, the spacelike $\widetilde{\Sigma}_s := t^{-1}(s)$ and the timelike $\mathcal{N}$ automatically intersect transversally. Hence, if $\Sigma_s := \widetilde{\Sigma}_s \cap \mathcal{N}$ is nonempty, it has to be a submanifold of $\widetilde{\mathcal{M}}$ of codimension 2 and a spacelike hypersurface of $\mathcal{N}$. In particular, such Σ_s is a Cauchy surface for $\mathcal{N}$: Let γ be a future directed inextendable causal curve in $\mathcal{N}$. Since $\mathcal{N}$ is closed, γ has to be defined on an open interval, say $\gamma \colon (a,b) \to \mathcal{N}$. Then, γ is also a future directed causal curve in $\widetilde{\mathcal{M}}$. Assume that γ is extensible as a future directed causal curve in $\widetilde{\mathcal{M}}$, w.l.o.g. let $\widetilde{\gamma} \colon (a,b] \to \widetilde{\mathcal{M}}$. Since $\mathcal{N}$ is closed, $\widetilde{\gamma}(b) \in \mathcal{N}$ which gives the contradiction. Hence, $J_-^{\mathcal{N}}(\Sigma_s) = t^{-1}(-\infty, s] \cap \mathcal{N}$ if Σ_s is nonempty. At least for some $u \in \mathbb{R}$ there has to be a point $p \in \Sigma_u$ since $\mathcal{N}$ is nonempty. The arguments from above show that any future directed inextendable causal curve γ in $\mathcal{N}$ through p is already a future directed inextendable causal curve in $\widetilde{\mathcal{M}}$ and, hence, intersects all $\widetilde{\Sigma}_s$. Thus, Σ_s is a Cauchy surface of $\mathcal{N}$ for all $s \in \mathbb{R}$ and $t|_{\mathcal{N}} \colon \mathcal{N} \to \mathbb{R}$ is a Cauchy time time function of $\mathcal{N}$. Analogously one sees that $t^{-1}(s) \cap \overline{\mathcal{M}}_i$ is a Cauchy surface of $\overline{\mathcal{M}}_i$. $\qquad\square$

2.2. **Spinorial preliminaries.** In the following we always assume that the spin structure is fixed. We denote by $S\mathcal{M}$ the associated *spinor bundle*, that is in particular a complex vector bundle with $N := 2^{\lfloor \frac{n+1}{2} \rfloor}$-dimensional fibers, denoted by $S_p\mathcal{M}$ for $p \in \mathcal{M}$, fiberwise endowed with a hermitian product

$$\langle \cdot \mid \cdot \rangle \colon S_p\mathcal{M} \times S_p\mathcal{M} \to \mathbb{C}$$

and with a *Clifford multiplication* $\gamma \colon T\mathcal{M} \to \mathrm{End}(S\mathcal{M})$ that is in particular fiber-preserving and satisfies for all $p \in \mathcal{M}$ and $u, v \in T_p\mathcal{M}$ that

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u,v)\mathrm{Id}_{S_p\mathcal{M}}.$$

We denote the by $\Gamma_c(\cdot)$, $\Gamma_{cc}(\cdot)$, $\Gamma_{sc}(\cdot)$ resp. $\Gamma(\cdot)$ the spaces of compactly supported, compactly supported in the interior, spacelike compactly supported resp. smooth sections of a vector bundle. The *(classical) Dirac operator* $\mathsf{D} \colon \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ is defined as the composition of the connection ∇ on $S\mathcal{M}$, obtained as a lift of the Levi-Civita connection on $T\mathcal{M}$, and the Clifford multiplication times $\imath$. Thus, in local coordinates this reads as

$$\mathsf{D} = \sum_{\mu=0}^{n} \imath \varepsilon_\mu \gamma(e_\mu) \nabla_{e_\mu}$$

where $(e_\mu)_{\mu=0,\ldots,n}$ is a local Lorentzian-orthonormal frame of $T\mathcal{M}$, $\varepsilon_\mu = g(e_\mu, e_\mu) = \pm 1$.

We recall that after the choice of the Cauchy time function t the metric on the globally hyperbolic manifold $(\widetilde{\mathcal{M}}, g)$ (without boundary) can be written as $g = g|_{\Sigma_t} - \beta^2 dt^2$ where $\beta \colon \widetilde{\mathcal{M}} \to \mathbb{R}$ is a positive smooth function [11]. This representation we will also use for $\bar{M}_i \subset \widetilde{M}$ as in Theorem 2.2. Thus, although the local orthonormal frame e_μ in general only exists locally, we have a globally defined unit timelike vector field

$$e_0 := \frac{1}{\beta} \partial_t \tag{2.1}$$

that we will use in the following. In particular we have $\gamma(e_0)^{-1} = \gamma(e_0)$.

2.3. **Reformulation as a symmetric positive hyperbolic system.** In this section we will formulate the Dirac equation (1.1) locally as a symmetric positive hyperbolic system. For that we shortly recall the basic definition, for more details see [34, 35].

For the following definition, let $E \to \mathcal{M}$ be a real (or complex) vector bundle with finite rank N endowed with the canonical fiberwise metric $\langle \cdot | \cdot \rangle := \langle \cdot | \cdot \rangle_{\mathbb{R}^N}$ (or $\langle \cdot | \cdot \rangle := \langle \cdot | \cdot \rangle_{\mathbb{C}^N}$). Moreover, let us endow $\Gamma_{cc}(E)$ with the L^2-scalar product

$$(\cdot | \cdot)_{\mathcal{M}} := \int_{\mathcal{M}} \langle \cdot | \cdot \rangle \mathrm{Vol}_{\mathcal{M}},$$

where $\mathrm{Vol}_{\mathcal{M}}$ denotes the volume element. Moreover, let $\|.\|_{L^2(\mathcal{M})}^2 := (. | .)_{\mathcal{M}}$.

Definition 2.4. A linear differential operator $\mathfrak{L} \colon \Gamma(E) \to \Gamma(E)$ of first order is called a *symmetric system* over $\mathcal{M}$ if

(S) the principal symbol $\sigma_{\mathfrak{L}}(\xi) \colon E_p \to E_p$ is hermitian with respect to $\langle \cdot | \cdot \rangle$ for every $\xi \in T_p^* \mathcal{M}$ and for every $p \in \mathcal{M}$.

Additionally, we say that $\mathfrak{L}$ is *positive* respectively *hyperbolic* if it holds:

(P) The bilinear form $\langle (\mathfrak{L} + \mathfrak{L}^\dagger) \cdot | \cdot \rangle$ on E_p is positive definite, where $\mathfrak{L}^\dagger$ denotes the formal adjoint of $\mathfrak{L}$ with respect to the L^2-product on $\Gamma_{cc}(E)$ and $\Re$ denotes the real part.

(H) For every future-directed timelike covector $\tau \in T_p^* \mathcal{M}$, the bilinear form $\langle \sigma_{\mathfrak{L}}(\tau) \cdot | \cdot \rangle$ is positive definite on E_p.

Let us recall that for a first-order linear operator $\mathfrak{L} \colon \Gamma(E) \to \Gamma(E)$ the principal symbol $\sigma_{\mathfrak{L}} \colon T^* \mathcal{M} \to \mathrm{End}(E)$ can be characterized by $\mathfrak{L}(fu) = f\mathfrak{L}u + \sigma_{\mathfrak{L}}(df)u$ where $u \in \Gamma(E)$ and $f \in C^\infty(\mathcal{M})$. If we choose local coordinates $(t, x^1, \ldots, x^n)$ on $\mathcal{M}$, with x^i local coordinates on Σ_t, and a local trivialization of E, any linear differential operator $\mathfrak{L} \colon \Gamma(E) \to \Gamma(E)$ of first order reads in a point $p \in \mathcal{M}$ as

$$\mathfrak{L} := A_0(p)\partial_t + \sum_{j=1}^n A_j(p)\partial_{x^j} + B(p)$$

where the coefficients A_0, A_j, B are $N \times N$ matrices, with N being the rank of E, depending smoothly on $p \in \mathcal{M}$. In these coordinates, Condition (S) in Definition 2.4 reduces to

$$A_0 = A_0^\dagger \qquad \text{and} \qquad A_j = A_j^\dagger$$

for $j = 1, \ldots, n$. Condition (P) reads as

$$\kappa := \mathfrak{L} + \mathfrak{L}^\dagger = B + B^\dagger - \frac{\partial_t(\sqrt{g}A_0)}{\sqrt{g}} - \sum_{j=1}^n \frac{\partial_{x^j}(\sqrt{g}A_j)}{\sqrt{g}} > 0, \qquad (2.2)$$

where g is the absolute value of the determinant of the Lorentzian metric. Condition (H) can be stated as follows: For any future directed, timelike covector $\tau = dt + \sum_j \alpha_j dx^j$,

$$\sigma_{\mathfrak{L}}(\tau) = A_0 + \sum_{j=1}^n \alpha_j A_j \quad \text{is positive definite.}$$

Remark 2.5. With the above definition, we can immediately notice that the Dirac operator does not give rise to a symmetric system. Consider for example the half

Minkowski spacetime $\mathscr{M}^4 := \mathbb{R}^3 \times [0, \infty)$ endowed with the standard metric $-dt^2 + dx^2 + dy^2 + dz^2$. In this setting the Dirac operator reads as

$$\mathsf{D} = -\imath\gamma(e_0)\partial_t + \imath\gamma(e_1)\partial_x + \imath\gamma(e_2)\partial_y + \imath\gamma(e_3)\partial_z$$

where $\gamma(e_i)$ are the Dirac matrices. By straightforward computation, we obtain

- $(\imath\gamma(e_j))^\dagger = \imath\gamma(e_j)$ for $j \geq 1$ while $(-\imath\gamma(e_0))^\dagger = \imath\gamma(e_0)$ which violates condition (S);
- $\kappa = 0$ and hence condition (P) is violated;
- $\sigma_\mathsf{D}(dt) = -\imath\gamma(e_0)$ is not positive definite, therefore condition (H) is violated.

Nonetheless, it still possible to find a fiberwise invertible endomorphism $Q \in \Gamma(\mathrm{End}(S\mathcal{M}))$ such that $Q \circ \mathsf{D}$ is a symmetric hyperbolic system such that for any compact subset of $\mathcal{M}$ the operator $Q \circ \mathsf{D} + \lambda\mathrm{Id}$ is also a positive hyperbolic system for a suitable $\lambda > 0$ as we will see below.

Lemma 2.6. *Consider a globally hyperbolic spin manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. Let D be the Dirac operator and e_0, β as in* (2.1). *Then for $\lambda \in \mathbb{R}$ the first order differential operator $\mathfrak{S}_\lambda \colon \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ defined by*

$$\mathfrak{S}_\lambda := \imath\gamma(e_0)\beta\mathsf{D} + \lambda\,\mathrm{Id} \tag{2.3}$$

is a symmetric hyperbolic system and its Cauchy problem

$$\begin{cases} \mathfrak{S}_\lambda\Psi = \mathfrak{f} \in \Gamma_c(S\mathcal{M}) \\ \Psi|_{\Sigma_0} = \mathfrak{h} \in \Gamma_c(S\Sigma_0) \\ \mathsf{M}\Psi|_{\partial\mathcal{M}} = 0 \end{cases} \tag{2.4}$$

is equivalent to the Cauchy problem for the Dirac operator

$$\begin{cases} \mathsf{D}\psi = f \in \Gamma_c(S\mathcal{M}) \\ \psi|_{\Sigma_0} = h \in \Gamma_c(S\Sigma_0) \\ \mathsf{M}\psi|_{\partial\mathcal{M}} = 0. \end{cases} \tag{2.5}$$

Moreover, for any compact set $\mathcal{R} \subset \mathcal{M}$, there exists a $\lambda > 0$ such that $\mathfrak{S}_\lambda$ is a symmetric positive hyperbolic system.

Proof. Since $\imath\gamma(e_0)\beta\mathsf{D}$ is a symmetric hyperbolic system, see e.g. [45, Chapter 3], $\mathfrak{S}_\lambda$ is symmetric and hyperbolic for any $\lambda \in \mathbb{R}$ and any positive smooth function β. Next we verify that the Cauchy problem for $\mathfrak{S}_\lambda$ and for D are equivalent for any $\lambda \in \mathbb{R}$. We set $\Psi := e^{-\lambda t}\psi$, which implies $\mathfrak{h} := e^{-\lambda t}h$, and $\mathfrak{f} := \imath e^{-\lambda t}\beta\gamma(e_0)f$ and obtain using (2.1)

$$\mathfrak{S}_\lambda\Psi = \mathfrak{S}_\lambda(e^{-\lambda t}\psi) = (\imath\gamma(e_0)\beta\mathsf{D} + \lambda\mathrm{Id})(e^{-\lambda t}\psi) = \imath e^{-\lambda t}\gamma(e_0)\beta\mathsf{D}\psi = \imath e^{-\lambda t}\beta\gamma(e_0)f.$$

Moreover, since $e^{-\lambda t} \neq 0$ for all $t \in \mathbb{R}$, we obtain the inverse map $(\mathfrak{f}, \mathfrak{h}) \mapsto (f, h)$ and we have

$$\mathsf{M}\Psi|_{\partial\mathcal{M}} = e^{\imath\lambda t}\mathsf{M}\psi|_{\partial\mathcal{M}} = 0 \quad \text{if and only if} \quad \mathsf{M}\psi|_{\partial\mathcal{M}} = 0.$$

This gives the equivalence of the two problems.

Let now $\mathcal{R} \subset \mathcal{M}$ be compact. It remains to check the positivity condition (P) for $\mathfrak{S}_\lambda$. The operator κ, defined in (2.2) for an arbitrary first order operator, is a zero order operator. Therefore, for any compact set $\mathcal{R}$, $\kappa|_{\Gamma(S\mathcal{R})}$ is bounded and there exists a suitable λ such that κ is positive definite on $\mathcal{R}$. $\qquad\square$
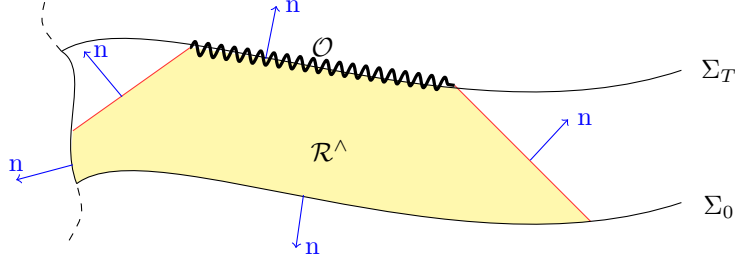
FIGURE 1. The set $\mathcal{R}^\wedge = J^-(\mathcal{O}) \cap \mathcal{T}$.

Example 2.7. Let us consider the half Minkowski spacetime $\mathscr{M}^4$ and the Dirac operator

$$\mathsf{D} = -\imath\gamma(e_0)\partial_t + \imath\gamma(e_1)\partial_x + \imath\gamma(e_2)\partial_y + \imath\gamma(e_3)\partial_z$$

from Remark 2.5. We can see that the operator

$$\mathfrak{S}_\lambda = \imath\gamma(e_0)\mathsf{D} + \lambda\mathrm{Id} = \partial_t - \gamma(e_0)\gamma(e_1)\partial_x - \gamma(e_0)\gamma(e_2)\partial_y - \gamma(e_0)\gamma(e_3)\partial_z + \lambda\mathrm{Id}$$

is a *symmetric positive hyperbolic system* for any $\lambda > 0$ on account of

$$\big(\gamma(e_0)\gamma(e_i)\big)^\dagger = \gamma(e_i)^\dagger\gamma(e_0)^\dagger = -\gamma(e_i)\gamma(e_0) = \gamma(e_0)\gamma(e_i) \quad \text{for } i = 1, 2, 3, \text{ and}$$

$\kappa = \mathfrak{S}_\lambda + \mathfrak{S}_\lambda^\dagger = 2\lambda\mathrm{Id}_{4\times4}$ and $\sigma_{\mathfrak{S}_\lambda}(dt) = \mathrm{Id}_{4\times4}$.

Remark 2.8. We note that one does not necessarily has to work with symmetric *positive* hyperbolic systems but symmetric hyperbolic systems probably would suffice if one uses a slightly different approach. We decided to require positivity since we go for a energy inequality as in Lemma 3.1 which is easy to obtain and enough for our purpose. One probably also can obtain energy inequalities as in [3, Thm. 5.3] without the positivity. The positivity requirement is also the reason why we work mostly on time strips and/or compact subsets. But at the end by uniqueness of solutions we will still obtain a global solution in Section 4.

3. LOCAL WELL-POSEDNESS OF THE CAUCHY PROBLEM

Let D be the Dirac operator on our globally hyperbolic spin manifold with timelike boundary $\partial\mathcal{M}$. We denote by $\mathcal{T}$ the *time strip* given by

$$\mathcal{T} := t^{-1}([0, T])$$

where $T > 0$ and $t\colon \mathcal{M} \to \mathbb{R}$ is the chosen Cauchy time function (For negative times see Remark 4.1). Let $\mathcal{O}$ be a compact subset of Σ_T. Let $\lambda > 0$ be such that the operator $\mathfrak{S}_\lambda\colon \Gamma(S\mathcal{M}) \to \Gamma(S\mathcal{M})$ defined as in (2.3) is a symmetric positive hyperbolic system on

$$\mathcal{R}^\wedge := J^-(\mathcal{O}) \cap \mathcal{T}$$

where $J^-(\mathcal{O})$ (resp. $J^+(\mathcal{O})$) denotes the *past set* (resp. *future set*), namely the set of all points that can be reached by past-directed resp. future-directed causal curves emanating from a point in $\mathcal{O}$, cp. Figure 1. This is always possible by Lemma 2.6 and since $\mathcal{R}^\wedge \subset \mathcal{M}$ is compact. For the reason why we choose $\mathcal{R}^\wedge$ as above compare below—especially Theorem 3.20.

In order to show existence of weak solutions and uniqueness of strong solutions for the Dirac Cauchy problem (2.5), we first shall derive so-called *"energy inequalities"*. These estimates have a clear physical consequence as we shall see in Proposition 3.4: Any solution can propagate with at most speed of light.

3.1. **Energy inequalities.** We note that our energy inequality (even when considered without boundary) differs from the one considered in [3, Thm. 5.3], see also Remark 2.8. Such an energy inequality therein should also be obtainable in our setting, but since our approach to well-posedness goes over positive symmetric hyperbolic systems the following inequality serves our purpose.

Lemma 3.1. *Let $\mathcal{O}$ be a compact subset of Σ_T and choose λ such that $\mathfrak{S}_\lambda$ is a symmetric positive hyperbolic system on $\mathcal{R}^\wedge$ as above. Then there exists a constant $c > 0$ such that for all $\Psi \in \Gamma(S\mathcal{T})$ satisfying $\Psi|_{\Sigma_0} = 0$ and $\mathsf{M}\Psi|_{\partial\mathcal{M}} = 0$ the energy inequality*

$$\|\Psi\|_{L^2(\mathcal{R}^\wedge)} \le c\|\mathfrak{S}_\lambda\Psi\|_{L^2(\mathcal{R}^\wedge)} \tag{3.1}$$

holds.

Proof. We first derive the Green formula for $\mathfrak{S}_\lambda$. In a local orthonormal frame e_μ we have $\mathfrak{S}_\lambda - \lambda = \sum_\mu \imath\gamma(e_0)\beta\imath\varepsilon_\mu\gamma(e_\mu)\nabla_{e_\mu} = -\sum_\mu \gamma(e_0)\beta\varepsilon_\mu\gamma(e_\mu)\nabla_{e_\mu}$. In a point of $\mathcal{R}^\wedge$ we obtain using $\gamma(e_0)^\dagger = \gamma(e_0)$ and $(\gamma(e_0)\gamma(e_j))^\dagger = \gamma(e_0)\gamma(e_j)$ that

$$\langle(\mathfrak{S}_\lambda - \lambda)\Psi, \Psi\rangle = \langle -\sum_\mu \beta\gamma(e_0)\varepsilon_\mu\gamma(e_\mu)\nabla_{e_\mu}\Psi, \Psi\rangle$$

$$= \langle(\nabla_{e_0} - \sum_j \gamma(e_0)\gamma(e_j)\nabla_{e_j})\Psi, \beta\Psi\rangle$$

$$= e_0\langle\Psi, \beta\Psi\rangle - e_j\langle\Psi, \beta\gamma(e_0)\gamma(e_j)\Psi\rangle + \langle\Psi, -\nabla_{e_0}(\beta\Psi) + \sum_j \nabla_{e_j}(\beta\gamma(e_0)\gamma(e_j)\Psi)\rangle$$

$$= \sum_\mu e_\mu\langle\Psi, -\beta\varepsilon_\mu\gamma(e_0)\gamma(e_\mu)\Psi\rangle + \langle\Psi, (\mathfrak{S}_\lambda^\dagger - \lambda)\Psi\rangle$$

where $\mathfrak{S}_\lambda^\dagger$ is the formal L^2-adjoint of $\mathfrak{S}_\lambda$. Let $\mathbf{n}$ be the outward normal vector to $\partial\mathcal{R}^\wedge$, cp. Figure 1. Thus, using Stokes formula and $\mathbf{n} = \sum_\mu \varepsilon_\mu g(\mathbf{n}, e_\mu)e_\mu$ we obtain

$$(\Psi \,|\, \mathfrak{S}_\lambda\Psi)_{\mathcal{R}^\wedge} - (\mathfrak{S}_\lambda^\dagger\Psi \,|\, \Psi)_{\mathcal{R}^\wedge} = (\Psi \,|\, \beta\gamma(e_0)\gamma(\mathbf{n})\Psi)_{\partial\mathcal{R}^\wedge} \tag{3.2}$$

where $(\cdot\,|\,\cdot)_{\partial\mathcal{R}^\wedge}$ is the induced L^2-product on $\partial\mathcal{R}^\wedge$. Subtracting $2(\Psi \,|\, \mathfrak{S}_\lambda\Psi)_{\mathcal{R}^\wedge}$, taking the real part and using that $\mathfrak{S}_\lambda$ is a symmetric positive system we thus obtain

$$\Re(\Psi \,|\, \beta\gamma(e_0)\gamma(\mathbf{n})\Psi)_{\partial\mathcal{R}^\wedge} - 2\Re(\Psi \,|\, \mathfrak{S}_\lambda\Psi)_{\mathcal{R}^\wedge} = -(\Psi \,|\, (\mathfrak{S}_\lambda + \mathfrak{S}_\lambda^\dagger)\Psi)_{\mathcal{R}^\wedge}$$
$$\le -2c(\Psi \,|\, \Psi)_{\mathcal{R}^\wedge}, \tag{3.3}$$

for some $c > 0$ and where $\Re$ denotes the real part. Next, let us decompose the boundary $\partial\mathcal{R}^\wedge$ as

$$\partial\mathcal{R}^\wedge = \mathcal{O} \cup \left(\Sigma_0 \cap J^-(\mathcal{O})\right) \cup Y,$$

where $Y := \partial J^-(\mathcal{O}) \cap \text{interior}(\mathcal{T})$ is the boundary of the light cone inside the time strip $\mathcal{T}$. The boundary term on $\Sigma_0 \cap J^-(\mathcal{O})$ vanishes by assumption on Ψ. Hence, we can have non zero boundary contributions only at $\mathcal{O}$ and Y. To deal with these terms, we decompose the boundary Y further as

$$Y = (Y \cap \partial\mathcal{M}) \sqcup \left(Y \setminus (Y \cap \partial\mathcal{M})\right).$$

Notice that if $\partial \mathcal{R}^\wedge \cap \partial \mathcal{M} = \emptyset$ then Y reduces to the boundary of the light cone $J^-(\mathcal{O})$ in the interior of $\mathcal{T}$. Imposing the boundary condition $\mathsf{M}\Psi|_{\partial \mathcal{M}} = 0$, we have by condition (1.2) that

$$(\Psi \,|\, \beta\gamma(e_0)\gamma(\mathbf{n})\Psi)_{\partial \mathcal{M}} = 0 \,.$$

Therefore, the boundary term on $Y \cap \partial \mathcal{M}$ vanishes. Hence, (3.3) reduces to

$$2\Re(\Psi \,|\, \mathfrak{S}_\lambda \Psi)_{\mathcal{R}^\wedge} - 2c(\Psi \,|\, \Psi)_{\mathcal{R}^\wedge} \geq \Re(\Psi \,|\, \beta\gamma(e_0)\gamma(\mathbf{n})\Psi)_{\mathcal{O}} + \Re(\Psi \,|\, \beta\gamma(e_0)\gamma(\mathbf{n})\Psi)_{Y \setminus (Y \cap \partial \mathcal{M})}.$$

Let us remark that the right hand side of the latter equation is non negative definite: Indeed, since $\mathcal{O}$ is a spacelike hypersurface, $\mathbf{n}|_{\mathcal{O}}$ is future directed timelike and by hyperbolicity of $\mathfrak{S}_\lambda$, $\langle . \,|\, \beta\gamma(e_0)\gamma(\mathbf{n}). \rangle_q = \langle . \,|\, \sigma_{\mathfrak{S}_\lambda}(\mathbf{n}^\flat). \rangle_{\cdot q}$ is a positive inner product for all $q \in \mathcal{O}$. By continuity, also the contribution on $Y \setminus (Y \cap \partial \mathcal{M})$ is still positive semidefinite since $Y \setminus (Y \cap \partial \mathcal{M})$ is a lightlike hypersurface. Hence, together with Inequality (3.3) we obtain

$$2c(\Psi \,|\, \Psi)_{\mathcal{R}^\wedge} \leq 2\Re(\Psi \,|\, \mathfrak{S}_\lambda \Psi)_{\mathcal{R}^\wedge}.$$

Thus, for all $\Psi \in \Gamma(S\mathcal{T})$ with $\Psi|_{\Sigma_0} = 0$ and $\mathsf{M}\Psi|_{\partial \mathcal{M}} = 0$ we have using the Hölder inequality that

$$\|\Psi\|_{L^2(\mathcal{R}^\wedge)} \leq c^{-1} \|\mathfrak{S}_\lambda \Psi\|_{L^2(\mathcal{R}^\wedge)}. \qquad \square$$

We conclude this section by giving an energy inequality for the formal L^2-adjoint of $\mathfrak{S}_\lambda$ analogous to (3.1). For any compact subset $\mathcal{O}' \subset \Sigma_0$, we restrict to the set $\mathcal{R}^\vee$ defined by

$$\mathcal{R}^\vee := \mathcal{T} \cap J^+(\mathcal{O}') \,. \tag{3.4}$$

Using the same arguments as in the proof of Lemma 3.1, we can conclude an analog energy inequality for the adjoint. Before stating this analog, let us shortly remark on the adjoint boundary condition.

Remark 3.2. The formal adjoint $\mathfrak{S}_\lambda^\dagger$ of $\mathfrak{S}_\lambda$ is by definition given by

$$(\Phi \,|\, \mathfrak{S}_\lambda \Psi)_{S\mathcal{M}} = (\mathfrak{S}_\lambda^\dagger \Phi \,|\, \Psi)_{S\mathcal{M}}$$

for all $\Phi \in \Gamma_c(S\mathcal{M})$ and $\Psi \in \Gamma_c(S\mathcal{M})$. Let $\mathfrak{S}_\lambda^*$ be the adjoint of $\mathfrak{S}_\lambda$ where the domain of $\mathfrak{S}_\lambda$ is given by $\mathrm{dom}\,\mathfrak{S}_\lambda := \{\Phi \in \Gamma_c(S\mathcal{T}) \mid \mathsf{M}\Phi|_{\partial \mathcal{M}} = 0, \Phi|_{\Sigma_0} = 0\}$. By the Green identity (3.2) we have

$$\mathrm{dom}\,\mathfrak{S}_\lambda^* = \left\{ \Phi \in \mathrm{dom}\,(\mathfrak{S}_\lambda^\dagger)_{\max} \,\Big|\, \int_{(\partial \mathcal{M} \cap \mathcal{T}) \cup \Sigma_T} \langle \Phi \,|\, \beta\gamma(e_0)\gamma(\mathbf{n})\Psi \rangle = 0 \; \forall \Psi \in \mathrm{dom}\mathfrak{S}_\lambda \right\}$$

where $\mathrm{dom}\,(\mathfrak{S}_\lambda^\dagger)_{\max}$ is the maximal domain of $\mathfrak{S}_\lambda^\dagger$. Note that

$$\left\{ \Phi \in \Gamma_c(S\mathcal{T}) \,\Big|\, \Phi|_{\Sigma_T} = 0, \int_{\partial \mathcal{M} \cap \mathcal{T}} \langle \Phi \,|\, \beta\gamma(e_0)\gamma(\mathbf{n})\Psi \rangle = 0 \; \forall \Psi \in \mathrm{dom}\mathfrak{S}_\lambda \right\}$$

is a core of $\mathfrak{S}_\lambda^*$. We set $(1 - \mathsf{M}^\dagger) \colon S_q(\mathcal{T} \cap \partial \mathcal{M}) \to S_q(\mathcal{T} \cap \partial \mathcal{M})$ to be the orthogonal projection to

$$\{\Phi \in S_q(\mathcal{T} \cap \partial \mathcal{M}) \mid \langle \Phi \,|\, \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q = 0 \; \forall \Psi \in S_q(\mathcal{T} \cap \partial \mathcal{M}) \text{ with } \mathsf{M}\Psi = 0 \} \,.$$

We call $\mathsf{M}^\dagger$ the *adjoint boundary condition*, compare [4, Sect. 7.2] for the analog notations in the elliptic case. Note that as a matrix $M^\dagger$ is *not necessarily* the hermitian adjoint of M (M can be multiplied by any invertible complex operator to obtain the same boundary condition.). Using this notation, we see that

$$\{\Phi \in \Gamma_c(S\mathcal{T}) \mid \Phi|_{\Sigma_T} = 0, \; \mathsf{M}^\dagger \Phi = 0\}$$
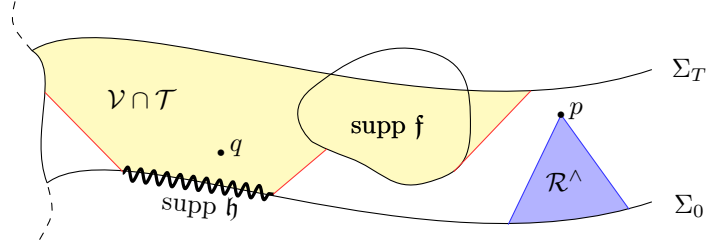
FIGURE 2. Finite propagation of speed – $\mathcal{V} \cap \mathcal{T}$.

is a core of $\mathfrak{S}_\lambda^*$.

Note that for $\mathsf{M} = \gamma(\mathbf{n}) - \imath$ being the MIT boundary condition then $\mathsf{M}^\dagger = \mathsf{M}$ — see also Remark 3.19.

Lemma 3.3. *Let $\Phi \in \Gamma(S\mathcal{T})$ satisfying $\Phi|_{\Sigma_T} = 0$ and $\mathsf{M}^\dagger\Phi|_{\partial\mathcal{M}} = 0$. Then Φ satisfies the inequality*

$$\|\Phi\|_{L^2(\mathcal{R}^\vee)} \leq C\|\mathfrak{S}_\lambda^\dagger\Phi\|_{L^2(\mathcal{R}^\vee)}. \tag{3.5}$$

3.2. **Uniqueness and finite propagation speed.** We are now ready to see by standard arguments that if there exists a smooth solution to the Cauchy problem (2.4), then it propagates with at most speed of light.

Proposition 3.4 (Finite speed of propagation). *Any smooth solution to the Dirac Cauchy problem* (1.1) *propagates with at most the speed of light, i.e., its support is inside the region*

$$\mathcal{V} := \Big(J^+\big(\mathrm{supp}\ f\big) \cup J^+\big(\mathrm{supp}\ h\big)\Big),$$

see Figure 2.

Proof. For any time strip $\mathcal{T}$, consider any point p outside the region $\mathcal{V} \cap \mathcal{T}$. Then there exists a λ such that $\mathfrak{S}_\lambda$ is a symmetric positive hyperbolic system on $\mathcal{R}^\wedge = \mathcal{T} \cap J^-(p)$. Let $\Psi = e^{-\lambda t}\psi$. Then, by Lemma 2.6 Ψ is a solution to (2.4) with $\mathfrak{f} = \imath e^{-\lambda t}\beta\gamma(e_0)f$ and $\mathfrak{h} = e^{-\lambda t}h$. By Lemma 3.1 and $\mathfrak{f}|_{\mathcal{R}^\wedge} \equiv 0$, $\mathfrak{h}|_{\mathcal{R}^\wedge \cap \Sigma_0} \equiv 0$, Ψ vanishes in $\mathcal{R}^\wedge$. Hence, Ψ vanishes outside $\mathcal{V}$.

The finite propagation of speed for a smooth solution of the Dirac Cauchy problem (2.5) then follows by Lemma 2.6. $\qquad\square$

Proposition 3.5 (Uniqueness). *Suppose there exist $\Psi, \Phi \in \Gamma(S\mathcal{T})$ satisfying the same Cauchy problem* (2.4) *for some $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ and $\mathfrak{h} \in \Gamma_{cc}(S\Sigma_0)$. Then $\Psi = \Phi$. In particular, this also gives uniqueness of smooth solutions for the Dirac Cauchy problem* (2.5).

Proof. Since Ψ and Φ satisfy the same initial-boundary value problem (2.4), then $\Psi - \Phi \in \Gamma(S\mathcal{T})$ is a solution of (2.4) with $\mathfrak{f} = 0$ and $\mathfrak{h} = 0$. By Proposition 3.4, the supports of Ψ and Φ are contained in $\mathcal{R}^\wedge$ for $\mathcal{O} := \mathcal{V} \cap \Sigma_T$. Therefore, we can use Lemma 3.1 to conclude that $\Psi - \Phi$ is zero.

The uniqueness of the Dirac Cauchy problem (2.5) then follows by Lemma 2.6. $\quad\square$

3.3. **Weak solutions in a time strip - definition.** In the following we will focus on the Cauchy problem (2.4) for $\mathfrak{h} \equiv 0$. This can be done without loss of generality, since:

Remark 3.6. For any Cauchy problem with any nonzero initial data $\mathfrak{h} \in \Gamma_{cc}(S\Sigma_0)$ there exists an equivalent Cauchy problem with zero initial data, namely

$$
\begin{cases}
\mathfrak{S}_\lambda \Psi = \mathfrak{f} \\
\Psi|_{\Sigma_0} = \mathfrak{h} \\
\mathsf{M}\Psi|_{\partial \mathcal{M}} = 0
\end{cases}
\iff
\begin{cases}
\mathfrak{S}_\lambda \widetilde{\Psi} = \widetilde{\mathfrak{f}} \\
\widetilde{\Psi}|_{\Sigma_0} = 0 \\
\mathsf{M}\widetilde{\Psi}|_{\partial \mathcal{M}} = 0
\end{cases}
$$

for any $\mathfrak{f}, \widetilde{\mathfrak{f}} \in \Gamma_c(S\mathcal{M})$. Here $\widetilde{\mathfrak{f}}(t,x) := \mathfrak{f}(t,x) - \lambda\mathfrak{h}(x) - \imath\beta\gamma(e_0)\mathsf{D}\mathfrak{h}$ and $\widetilde{\Psi}(t,x) := \Psi(t,x) - \mathfrak{h}(x)$. Note that $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ if and only if $f \in \Gamma_{cc}(S\mathcal{M})$.

With the help of the Energy inequality (3.1) we shall prove the existence of a *weak solution* for the mixed initial-boundary value problem (2.4) for $\mathfrak{S}_\lambda$ as in (2.3) (for fixed $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ and $\mathfrak{h} \equiv 0$). To this end, let λ be such that $\mathfrak{S}_\lambda^\dagger$ is a symmetric positive hyperbolic system on $\mathcal{R}^\vee = \mathcal{T} \cap J^+(\mathcal{O}')$ for

$$
\mathcal{O}' := J^-(\mathcal{V} \cap \Sigma_T) \cap \Sigma_0 \text{ with } \mathcal{V} := J^+\big(\text{supp } \mathfrak{f}\big) \cap \mathcal{T}. \tag{3.6}
$$

We denote by

$$
\mathscr{H} := \overline{\big(\Gamma_c(S\mathcal{T}), (\,.\,|\,.\,)_\mathcal{T}\big)}^{(\,.\,|\,.\,)_\mathcal{T}}
$$

the L^2-completion of $\Gamma_c(S\mathcal{T})$.

Definition 3.7. We call $\Psi \in \mathscr{H}$ a *weak solution* to the Cauchy problem (2.4) restricted to an open subset $\mathcal{U}$ of $\mathcal{T}$ with $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ and $\mathfrak{h} \equiv 0$ if the relation

$$
(\Phi \mid \mathfrak{f})_\mathcal{U} = (\mathfrak{S}_\lambda^\dagger \Phi \mid \Psi)_\mathcal{U} \tag{3.7}
$$

holds for all $\Phi \in \Gamma_c(S\mathcal{U})$ satisfying $\mathsf{M}^\dagger\Phi|_{\partial\mathcal{M}} = 0$ and $\Phi|_{\Sigma_T} \equiv 0$.

In order to check that this is the right definition let us give the following remark.

Remark 3.8. Let Ψ be a weak solution as defined above that is even smooth. Then by testing first only with $\Phi \in \Gamma_{cc}(S\mathcal{U})$ we immediately obtain with the Green identity that $\mathfrak{S}_\lambda \Psi = \mathfrak{f}$. Using this in (3.7) with the Green identity for general $\Phi \in \Gamma_c(S\mathcal{U})$ gives

$$
0 = (\Phi \mid \beta\gamma(e_0)\gamma(\mathbf{n})\Psi)_{(\text{interior}(\mathcal{T}) \cap \partial\mathcal{M}) \cup \Sigma_0 \cup \Sigma_T}.
$$

The part on Σ_T vanishes since $\Phi_{\Sigma_T} \equiv 0$. Moreover, using test functions Φ that have support near Σ_0 resp. $\partial\mathcal{M} \cap \text{interior}(\mathcal{T})$ one sees as in Remark 3.2 that $\Psi_{\Sigma_0} = 0$ resp. $\mathsf{M}\Psi|_{\partial\mathcal{M}} = 0$.

Remark 3.9. Analogously, one sees that a spinor $\Psi \in \mathscr{H}$ is a weak solution to (2.4) restricted to an open subset $\mathcal{U}$ of $\mathcal{T}$ with $\mathfrak{h} \in \Gamma_{cc}(S\Sigma_0)$ if the relation

$$
(\Phi \mid \mathfrak{f})_\mathcal{U} = (\mathfrak{S}_\lambda^\dagger \Phi \mid \Psi)_\mathcal{U} + (\Phi \mid \beta\gamma(e_0)\gamma(\mathbf{n})\mathfrak{h})_{\Sigma_0 \cap \mathcal{U}}
$$

holds for all $\Phi \in \Gamma_c(S\mathcal{U})$ satisfying $\mathsf{M}^\dagger\Phi|_{\partial\mathcal{M}} = 0$ and $\Phi|_{\Sigma_T} \equiv 0$.
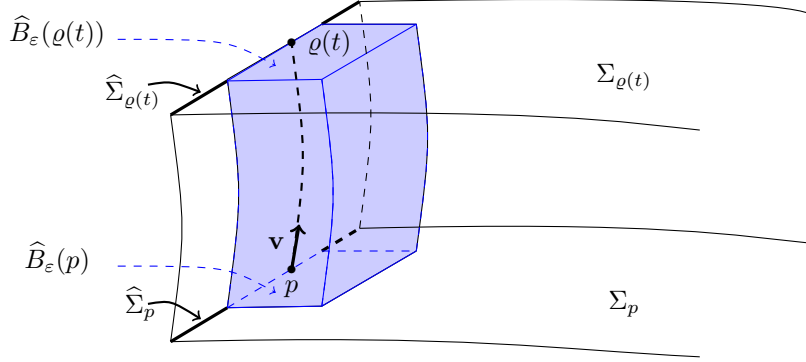
FIGURE 3. Fermi coordinates on each Cauchy surface.

3.4. **Differentiability of weak solutions.** Before examining existence of a weak solution, we want to show that a weak solution, if it exists, is a strong solution and, in particular, smooth. Since this is a local question, we use the theory for hyperbolic systems on subsets of $\mathbb{R}^{n+1}$—in particular that a weak solution is a (semi-)strong solution, [42, Section 1], and the regularity estimates for strong solutions in [47, Theorem 3.1]. Since the definition of strong solution in the sense Lax–Phillips slightly differs with the one given by Rauch–Massey, we will denote it by semi-strong in the following:

Definition 3.10. Let $U \subset \mathcal{M}$ be a compact subset in $\mathcal{M}$. We say that $\Psi \in \mathscr{H}$ is a *semi-strong solution* of the initial-boundary value problem (2.4) if there exists a sequence of sections $\Psi_k \in W^{1,2}(\Gamma(SU))$ such that $\mathsf{M}\Psi_k = 0$ on $\partial\mathcal{M} \cap U$, $\Psi_k = 0$ on Σ_0 and

$$\|\Psi_k - \Psi\|_{L^2(U)} \xrightarrow{k\to\infty} 0 \quad \text{and} \quad \|\mathfrak{S}_\lambda\Psi_k - \mathfrak{f}\|_{L^2(U)} \xrightarrow{k\to\infty} 0.$$

The solution is called *strong* if additionally the sequence Ψ_k can chosen to be smooth.

We concentrate on points in the boundary $p \in \partial\mathcal{M}$ (the other points will even be easier since we do not have to care about boundaries) and firstly define a convenient chart as follows, compare also Figure 3: Let Σ_p be the Cauchy surface of $\mathcal{M}$ to which p belongs to. For $q \in \partial\mathcal{M}$ let $\widehat{\Sigma}_q := \Sigma_q \cap \partial\mathcal{M}$ be the corresponding Cauchy surface in the boundary. Let $\varrho\colon [0,\varepsilon] \to \partial\mathcal{M}$ be the timelike geodesic in $\partial\mathcal{M}$ starting at p with velocity $\mathbf{v} \in T_p\partial\mathcal{M}$ where $\mathbf{v}$ is a normalized, future-directed, timelike vector perpendicular to $\widehat{\Sigma}_p$ in $\partial\mathcal{M}$. Let $\widehat{B}_\varepsilon(\varrho(t))$ be the ε-ball in $\widehat{\Sigma}_{\varrho(t)}$ around $\varrho(t)$. On these balls we choose geodesic normal coordinates $\widehat{\kappa}_t\colon B_\varepsilon^{n-1}(0) \subset \mathbb{R}^{n-1} \to \widehat{B}_\varepsilon(\varrho(t))$. Moreover, inside each $\Sigma_{\varrho(t)}$ we choose Fermi coordinates with base $\widehat{B}_\varepsilon(\varrho(t))$. Thus, we obtain a chart in $\Sigma_{\varrho(t)}$ around $\varrho(t)$ as

$$\widetilde{\kappa}_t\colon B_\varepsilon^{n-1}(0) \times [0,\varepsilon] \subset \mathbb{R}^n \to U_\varepsilon(\widehat{B}_\varepsilon(\varrho(t))) := \{q \in \Sigma_{\varrho(t)} \mid \mathrm{dist}_{\Sigma_{\varrho(t)}}(q, \widehat{B}_\varepsilon(\varrho(t)) \leq \varepsilon\}$$

$$(y, z) \mapsto \exp_{\widehat{\kappa}_t(y)}^{\perp, \Sigma_{\varrho(t)}}(z)$$

where $\exp_{\widehat{\kappa}_t(y)}^{\perp, \Sigma_{\varrho(t)}}(z)$ is the normal exponential map in $\Sigma_{\varrho(t)}$ starting at $\widehat{\kappa}_t(y)$ with velocity perpendicular to $\widehat{\Sigma}_{\varrho(t)} = \partial\Sigma_{\varrho(t)}$ pointing in the interior and with magnitude

z. Putting all this together we obtain a chart

$$\kappa_p \colon [0,\varepsilon] \times B^{n-1}_\varepsilon(0) \times [0,\varepsilon] \subset \mathbb{R}^{n+1} \to U_p := \bigcup_{t \in [0,\varepsilon]} U_\varepsilon(\widehat{B}_\varepsilon(\varrho(t))) \subset \mathcal{M}$$

$$(t, y, \bar{z}) \mapsto \widetilde{\kappa}_t(y, \bar{z}).$$

Note that sections of the spinor bundle SU_p are now just vector-valued functions $U_p \to \mathbb{C}^N$ where N is the rank of the spinor bundle.

For more details on the above geometric maps compare e.g. [39]. For us here, the only purpose of those charts is to specify coordinates such that near the point p the Cauchy problem is near enough to the Minkowski standard form and will take the form as in [42, 47]. To see this, let us first consider the model case of a 'general half' of the Minkowski space.

Example 3.11. Let $\overline{\mathcal{M}}$ be the Minkowski space with coordinates $x = (x^0, \ldots, x^n)$ with the standard foliation of Cauchy surfaces. We set $t := x^0$, $y = (x^1, \ldots, x^{n-1})$ and $z := x^n$. For $|a| < 1$, the hypersurface $\mathcal{N}_a := \{z = at\}$ is timelike and $\mathcal{M}_a = \{z \geq at\} \subset \overline{\mathcal{M}}$ is a globally hyperbolic manifold with a timelike boundary. We use $(\bar{t} := t, y, \bar{z} := z - at)$ as new coordinates on $\mathcal{M}_a$. Then, $(\bar{t}, y, \bar{z}) \mapsto (\bar{t}, y, \bar{z} + a\bar{t})$ is exactly the map κ_p from above for any $p \in \mathcal{N}_a$.

Then, together with $\gamma(e_0) = \gamma(e_0)^{-1}$, we have

$$\imath\gamma(e_0)\mathsf{D} = \partial_t - \sum_{j=1}^{n-1} \gamma(e_0)\gamma(e_j)\partial_{x^j} - \gamma(e_0)\gamma(e_n)\partial_z$$

$$= \partial_{\bar{t}} - \sum_{j=1}^{n-1} \gamma(e_0)\gamma(e_j)\partial_{x^j} - (\gamma(e_0)\gamma(e_n) + a)\partial_{\bar{z}}.$$

Thus, $\mathfrak{S}_\lambda = \imath\gamma(e_0)\mathsf{D} + \lambda\mathrm{Id}$, as in Example 2.7, has the form

$$\mathfrak{S}_\lambda = \partial_{\bar{t}} + \sum_{j=1}^{n-1} A_j(x)\partial_{x^j} + A_{\bar{z}}(x)\partial_{\bar{z}} + B(x). \tag{3.8}$$

with $A_{\bar{z}}(x) = -\gamma(e_0)\gamma(e_n) - a$. Since $|a| < 1$, $A_{\bar{z}}(x)$ is nonsingular on $\partial\mathcal{M}_a$. Hence, since $\ker \mathsf{M}|_q$ varies smoothly with $q \in \partial\mathcal{M}_a$, after restricting to some cube in $\mathcal{M}_a$ we are exactly in the situation considered in [42, 47].

Corollary 3.12. *For the Dirac operator D on a globally hyperbolic spin manifold with timelike boundary $\partial\mathcal{M}$ and $p \in \partial\mathcal{M}$, there is a sufficiently small $\varepsilon > 0$ such that in the coordinates κ_p from above there is an invertible operator $\mathfrak{E} \colon \Gamma(U_p, \mathbb{C}^N) \to \Gamma(U_p, \mathbb{C}^N)$ such that $\mathfrak{E}\mathfrak{S}_\lambda$ has the form*

$$\partial_{\bar{t}} + \sum_{j=1}^{n-1} A_j(\bar{t}, y, \bar{z})\partial_{x^j} + A_{\bar{z}}(\bar{t}, y, \bar{z})\partial_{\bar{z}} + B(\bar{t}, y, \bar{z})$$

with $A_{\bar{z}}$ nonsingular on the boundary. In particular, any weak solution of the Cauchy problem (2.4) gives rise to a weak solution to the Cauchy problem

$$\begin{cases} \widehat{\mathfrak{S}_\lambda} := \mathfrak{E}\mathfrak{S}_\lambda\Psi & = \mathfrak{f} \in \Gamma_c(SU_p) \\ \Psi|_{V_p} & = \mathfrak{h} \in \Gamma_c(SV_p) \\ \mathsf{M}\Psi|_{\partial\mathcal{M}} & = 0 \end{cases} \tag{3.9}$$

on U_p and vice versa. Here $V_p := U_p \cap t^{-1}(0)$ and weak solution of (3.9) is defined analogously as in Definition 3.7.

Proof. By the choice of the coordinates, the Dirac operator will look in p exactly as for the Minkowski space computed in Example 3.11 with $a = dt(n_p)$ where n_p is the normal at p of $\hat{\Sigma}_p$ in $\partial\mathcal{M}$ and t is the global time function of $\overline{\mathcal{M}}$. Note that the role of $\mathcal{N}_a$ is taken by the tangent plane of $\partial\mathcal{M} \hookrightarrow \mathcal{M}$ in p and that $|a| < \beta(p)^{-1}$. Since everything is continuous, we can find a sufficiently small $\varepsilon > 0$ such that there is an invertible linear map $\mathfrak{E} \colon \Gamma(U_p, \mathbb{C}^N) \to \Gamma(U_p, \mathbb{C}^N)$ with $\mathfrak{E}|_{S_p\mathcal{M}} = 1$ and such that $\mathfrak{E}\mathfrak{S}_\lambda$ has the required form with $A_{\overline{z}}(p) = -\beta\gamma(e_0)\gamma(e_n) - a$.
Moreover, we obtain a weak solution of (3.9) as required by taking the weak solution of (2.4) where the right handside is given by $(\mathfrak{E}^{-1}\mathfrak{f}, \mathfrak{h}, 0)$. $\qquad\square$

Lemma 3.13 (Locally strong solution)**.** *A weak solution Ψ of the Cauchy problem (3.9) is a strong solution on U_p.*

Proof. The last Corollary tells us that we can apply [42, Section 2] in order to obtain the existence of a semi-strong solution, i.e., there is a sequence of continuous sections $\Phi_k \in W^{1,2}(U_p, \mathbb{C}^N)$ with $\mathsf{M}\Phi_k|_{\partial\mathcal{M}\cap U_p} = 0$ and $\|\Phi_k - \Psi\|_{L^2(U_p)} \to 0$ and $\|\widehat{\mathfrak{S}}_\lambda \Phi_k - \mathfrak{f}\|_{L^2(U_p)} \to 0$. It remains to argue, that we can approximate the Φ_k by smooth Ψ_k still fulfilling the boundary condition and the convergences from above. This can be achieved using standard Sobolev theory; we refer to [27] for more details: First, choose $u_i \in \Gamma(U_p, \mathbb{C}^N)$, $i = 1, \ldots, r$, such that for each $q \in \partial\mathcal{M} \cap U_p$ they form a basis of $\ker \mathsf{M}|_q$ and are linearly independent in all $q \in U_p$. Since M depends smoothly on the base point and has constant rank this is always possible. Choose $u_j \in \Gamma(U_p, \mathbb{C}^N)$, $j = r+1, \ldots, N$ such that $u_1(q), \ldots, u_N(q)$ is a basis of $\mathbb{C}^N$ at each $q \in U_p$. A section $\Phi \in W^{1,2}(U_p, \mathbb{C}^N)$ can now be expressed as $\Phi = \sum_{i=1}^N a_i u_i$ for $a_i \colon U_p \to \mathbb{C}$. We denote by Φ^+ the part of Φ spanned by u_1 to u_r and set $\Phi^- := \Phi - \Phi^+$. Using the a_i as the new coordinates, we decompose the solution Ψ into $\Psi_k^+ \in W^{1,2}(U_p, \mathbb{C}^r)$ and $\Psi_k^- \in W^{1,2}(U_p, \mathbb{C}^{N-r})$. Thus, there is a sequence $\Psi_{k,j}^+ \in \Gamma(U_p, \mathbb{C}^r)$ that converges to Ψ_k^+ in $W^{1,2}$ and analogously a smooth sequence $\Psi_{k,j}^-$ converging to Ψ_k^- in $W^{1,2}$. Moreover, by definition $\mathtt{tr}\,\Psi_k^+ = 0$, where $\mathtt{tr}$ is the trace map $W^{1,2}(U_p, \mathbb{C}^r) \to L^2(U_p \cap \partial\mathcal{M}, \mathbb{C}^r)$. Thus, $\Psi_{k,j}^+$ can be chosen to be zero on $U_p \cap \partial\mathcal{M}$. Thus, $\Psi_{k,j} = \Psi_{k,j}^+ + \Psi_{k,j}^-$, where we use the embeddings $\mathbb{C}^r \hookrightarrow \mathbb{C}^N$ and $\mathbb{C}^r \hookrightarrow \mathbb{C}^{N-r}$ from above, are smooth sections fulfilling $\mathsf{M}\Psi_{k,j}|_{U_p \cap \partial\mathcal{M}} = 0$. Choosing a diagonal sequence we obtain smooth Ψ_k approximating Ψ as in Definition 3.10. $\qquad\square$

Next we want to see whether the strong solution on U_p is actually smooth. For that we would like to use the result [47, Theorem 3.1] by Rauch and Massey. For $p \in \Sigma_0$ this is immediate. But for $p \in \Sigma_t$ for $t > 0$ the solution might now touch the boundary and compatibility issues occur. This is expectable since just assuming $\mathfrak{f} \in \Gamma_c(S\mathcal{M})$ and $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ (and not as in Theorem 1.1 compactly supported in the interior) is not sufficient to guarantee that the solution of the Cauchy problem (2.4) is smooth.

3.4.1. *Compatibility conditions and smoothness of the solution.* To see the appearance of the compatibility issues let us start with the easiest example:

Example 3.14. Let $\mathcal{M}_a$ be the half Minkowski spacetime as described in Example 3.11 and consider the Cauchy problem (2.4) $\mathfrak{S}_\lambda \Psi = 0$, $\Psi|_{t=0} = \mathfrak{h}$ and

$\mathsf{M}\Psi|_{\overline{z}=0} = 0$. Assume that M does not depend on t, that is (e.g.) true for MIT boundary conditions. Suppose that Ψ is k-differentiable. Set $\mathfrak{G} := \partial_t - \mathfrak{S}_\lambda$. Then it satisfies

$$0 = \partial_t^k\big(\mathsf{M}\Psi|_{\overline{z}=0}\big)|_{t=0} = \big(\mathsf{M}\partial_t^k\Psi|_{\overline{z}=0}\big)|_{t=0} = \big(\mathsf{M}\mathfrak{G}^k\Psi(t)|_{\overline{z}=0}\big)|_{t=0} = \mathsf{M}\big(\mathfrak{G}^k\mathfrak{h}\big)|_{\overline{z}=0}\,.$$

Therefore, any initial data have to satisfy a compatibility condition.

Remark 3.15. In the previous example we massively used that M and $\mathfrak{G}$ does not depend on t. In the more general case of $\mathfrak{S}_\lambda$ for a Dirac operator on a globally hyperbolic manifold in the coordinates on U_p defined on page 14 this is in general not the case.

In order to obtain a general compatibility condition on $\mathfrak{f}$ and $\mathfrak{h}$ set $\mathfrak{G} = \partial_t - \mathfrak{E}\mathfrak{S}_\lambda$ with $\mathfrak{E}$ as in Corollary 3.12 and

$$\mathfrak{h}_k := \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}\big(\partial_t^j\mathfrak{G}\big)|_{V_p}\mathfrak{h}_{k-1-j} + \partial_t^k\mathfrak{f}|_{V_p} \tag{3.10}$$

for all $k \geq 1$ with $\mathfrak{h}_0 = \mathfrak{h}$, $V_p = U_p \cap \Sigma_0$. Impose that the data $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ and $\mathfrak{f} \in \Gamma(S\mathcal{M})$ satisfy

$$\sum_{j=1}^{k} \frac{k!}{(k-j)!}\big(\partial_t^j\mathsf{M}\big)|_{V_p}\mathfrak{h}_{j-1} = 0.$$

Translating this back for our Dirac Cauchy problem (2.5) in the Hamiltonian form where $\mathsf{H} := \partial_t - \imath\beta\gamma(e_0)\mathsf{D}$

$$(\partial_t - \mathsf{H})\psi = \imath\beta\gamma(e_0)f$$

the compatibility condition for $h \in \Gamma_c(S\Sigma_0)$ and $\mathfrak{f} \in \Gamma(S\mathcal{M})$ reduces to

$$\sum_{j=1}^{k} \frac{(k)!}{j!(k-j)!}\big(\partial_t^j\mathsf{M}\big)\Big|_{\partial\Sigma_0} h_{k-1} = 0$$

for all $k \geq 1$ where

$$h_k := \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}(\partial_t^j\mathsf{H})|_{\partial\Sigma_0}\, h_{k-1-j} + \partial_t^p\big(-\imath\beta\gamma(e_0)f\big)|_{\partial\Sigma_0}$$

with $h_0 = h$.

With the above definitions the following corollary follows directly by localizing a solution in any set U_p defined as above and then applying [47, Theorem 3.1, see also p. 305].

Corollary 3.16 (Local smooth solution)**.** *Let U_p as above. Then a strong solution for the Cauchy problem (2.4) is smooth on U_p if and only if $\mathfrak{h} \in \Gamma_c(U_p, \mathbb{C}^N)$ and $\mathfrak{f} \in \Gamma(U_p, \mathbb{C}^N)$ satisfy*

$$\sum_{j=1}^{k} \frac{k!}{(k-j)!}(\partial_t^j\mathsf{M})|_{\partial\Sigma_0}\,\mathfrak{h}_{k-1} = 0 \tag{3.11}$$

with $\mathfrak{h}_i$ as in (3.10).

Remark 3.17. If we choose initial data $\mathfrak{h}$ and $\mathfrak{f}$ with compact support in the interior of Σ_0 resp. $\mathcal{M}$, the compatibility condition is automatically satisfied. Actually for $\mathfrak{f}$ it is enough to be zero in a neighborhood of $\partial\Sigma_0 \subset \mathcal{M}$.

Since the compatibility equation (3.11) is an 'if and only if'-criterion for smoothness, we can use iteratively already obtained smoothness for U_p with $t(p) < s$ to obtain that the initial data on U_q with $t(p) = s$ fulfills the compatibility criterion as well. That way we will obtain smoothness on the full time strip:

Corollary 3.18. *A weak solution of the Cauchy problem* (3.9) *for* $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ *and* $\mathfrak{h} \in \Gamma_{cc}(S\Sigma_0)$ *in an open subset* $\mathcal{U}$ *of* $\mathcal{T}$ *is smooth. In particular, there is a smooth solution of the Dirac Cauchy problem* (1.1) *in* $\mathcal{U}$.
The above statement remains true for all $\mathfrak{f} \in \Gamma_c(S\mathcal{M})$ *and* $\mathfrak{h} \in \Gamma_c(S\Sigma_0)$ *fulfilling* (3.11).

Proof. First let $p \in \partial\mathcal{M} \cap \Sigma_{\hat{t}}$ for some $\hat{t} \in [0,T]$ and let $\varrho \colon [0,\hat{t}] \to \partial M$ be a timelike curve with $\varrho(0) \in \Sigma_0$ and $p = \varrho(\hat{t})$. We fix $\varepsilon > 0$ such that we have Fermi coordinates on a 'cube' $U_{\varrho(t)}$ around $\varrho(t)$ as in Section 3.4 for all $t \in [0,\hat{t}]$ and such that Corollary 3.12 holds for those cubes. This is always possible since the image of ϱ is compact and everything depends smoothly on the basepoints.
For $U_{\varrho(0)}$ we know that the compatibility condition (3.11) is fulfilled by assumption. Thus, Corollary 3.16 tells us that the weak solution Ψ is smooth in $U_{\varrho(0)}$ and that for every $a \in [0,\varepsilon]$ the function $\mathfrak{h}_a := \Psi|_{U_{\varrho(0)} \cap \Sigma_a}$ as new initial data $\mathfrak{h}$ together with the original $\mathfrak{f}$ still fulfill the compatibility condition. Moreover, $\Psi|_{U_{\varrho(a)}}$ is still a weak solution to the initial data $(\mathfrak{h}_a, \mathfrak{f})$ on $U_{\varrho(a)}$. Thus, we can again use Corollary 3.16 where Σ_a now takes the role of Σ_0. Iterating this procedure, we obtain smoothness on all $U_{\varrho(t)}$ for $t \in [0,\hat{t}]$, i.e. in particular in p.
For $p \in \mathcal{M} \setminus \partial\mathcal{M}$ we choose a timelike curve $\varrho \colon [0,\hat{t}] \to \mathcal{M} \setminus \partial\mathcal{M}$ with $\varrho(0) \in \Sigma_0$ and $p = \varrho(T)$ and proceed as before. It is even easier since we can just use geodesic normal coordinates in the Cauchy surfaces around each $\rho(t)$. The existence of smooth solutions to the Dirac Cauchy problem (2.5) then follows by Lemma 2.6. $\square$

Remark 3.19. In view of Remark 1.2, we want to comment on the assumptions on the boundary condition that we have used up to here. The energy inequalities (3.1) and (3.5) need that $\mathsf{M}\Psi|_{\partial\mathcal{M}} = 0$ and $\mathsf{M}^\dagger\Psi|_{\partial\mathcal{M}} = 0$ both imply

$$\langle \Psi \mid \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q = 0$$

for all $q \in \partial\mathcal{M}$. Moreover, in order to apply [42] in Lemma 3.13 and [47] in Corollary 3.16, we additionally use that $\ker \mathsf{M}|_q$ is nonempty and varies smoothly with $q \in \partial\mathcal{M}$.
The properties collected above are valid for the MIT bag boundary condition $\mathsf{M} := \gamma(\mathbf{n}) - \imath$ (and analogously for $\mathsf{M} := \gamma(\mathbf{n}) + \imath$). Indeed, on account of

$$\langle \Psi \mid \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q = \langle \gamma(e_0)\Psi \mid \gamma(\mathbf{n})\Psi \rangle_q = \langle -\gamma(\mathbf{n})\gamma(e_0)\Psi \mid \Psi \rangle_q$$
$$= \langle \gamma(e_0)\gamma(\mathbf{n})\Psi \mid \Psi \rangle_q = \overline{\langle \Psi \mid \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q},$$

where in the third equality we used $g(e_0, \mathbf{n}) = 0$, we obtain $\langle \Psi \mid \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q \in \mathbb{R}$. By using MIT boundary conditions, we have

$$\langle \Psi \mid \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q = \langle \Psi \mid \gamma(e_0)\imath\Psi \rangle_q = \langle \gamma(e_0)\Psi \mid \imath\Psi \rangle_q =$$
$$= \langle -\gamma(e_0)\imath\Psi \mid \Psi \rangle_q = \langle -\gamma(e_0)\gamma(\mathbf{n})\Psi \mid \Psi \rangle_q = -\overline{\langle \Psi \mid \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q},$$

which implies $\langle \Psi \mid \gamma(e_0)\gamma(\mathbf{n})\Psi \rangle_q = 0$ for $\mathsf{M}\Psi = 0$. The rest follows since $\mathsf{M} = \mathsf{M}^\dagger$ as we will see in the following: Note that this in particular implies that H as in

Remark 3.15 restricted to a fixed Cauchy surface Σ_t is essentially self-adjoint. First we rewrite the boundary condition $\mathsf{M}\Psi = 0$ as

$$P_+\Psi := \frac{1}{2}(\mathrm{Id} + \imath\gamma(\mathbf{n}))\Psi = 0$$

and we set $P_-\Psi := \frac{1}{2}(\mathrm{Id} - \imath\gamma(\mathbf{n}))\Psi = 0$. It is easy to see that

$$P_+ + P_- = \mathrm{Id} \qquad P_\pm^2 = P_\pm \qquad \text{and} \qquad P_+P_- = P_-P_+$$

and in particular that $\langle P_+\Psi \,|\, P_-\Phi\rangle_q = 0$ for all $\Psi, \Phi \in S_q\mathcal{M}$. This implies that $P_+\Psi = 0$ is tantamount to set $\Psi = P_-\widetilde{\Psi}$ for some $\widetilde{\Psi}$. Thus, for Ψ verifying $\mathsf{M}\Psi = 0$, the condition $\langle\Phi \,|\, \gamma(e_0)\gamma(\mathbf{n})\Psi\rangle_q = 0$ holds if and only if for any $\widetilde{\Psi}$ it is satisfied

$$0 = 2\langle\Phi \,|\, \gamma(e_0)\gamma(\mathbf{n})P_-\widetilde{\Psi}\rangle_q = \langle\gamma(e_0)\Phi \,|\, (\gamma(\mathbf{n}) + \imath)\widetilde{\Psi}\rangle_q =$$
$$= \langle(-\gamma(\mathbf{n}) - \imath)\gamma(e_0)\Psi \,|\, \widetilde{\Phi}\rangle_q = \langle\gamma(e_0)(\gamma(\mathbf{n}) - \imath)\Phi \,|\, \widetilde{\Psi}\rangle_q \,,$$

which implies $\mathsf{M}\Phi = (\gamma(\mathbf{n}) - \imath)\Phi = 0$.

Another example is the chirality operator $\mathsf{M} := (\mathrm{Id} - \gamma(\mathbf{n})\mathcal{G})$ where $\mathcal{G}$ is the restriction to $\partial\mathcal{M}$ of an endomorphism-field of $S\mathcal{M}$ which is involutive, unitary, parallel and anti-commuting with the Clifford multiplication on $\mathcal{M}$, [38, Section 1.5].

3.5. **Existence of a weak solution.** Up to now we have seen in Corollary 3.18, Proposition 3.4 and Proposition 3.5, that if there exists a weak solution Ψ of the Cauchy problem (2.4), compare Definition 3.7, then it is smooth, unique and it vanishes outside $\mathcal{V}$ defined by (3.6). It just remains to prove existence.

Theorem 3.20 (Weak existence). *There exists a unique weak solution $\Psi \in \mathscr{H}$ to the Cauchy problem (2.4) with $\mathfrak{f} \in \Gamma_{cc}(S\mathcal{M})$ and $\mathfrak{h} \equiv 0$, restricted to $\mathcal{T}$.*

Proof. Let $\mathcal{U}$ be a compact subset with supp $\mathfrak{f} \Subset \mathcal{U}$ and set $\mathcal{V} := J^+(\mathrm{supp}\,\mathfrak{f}) \cap \mathcal{T}$ and $\mathcal{V}' := J^+(\mathcal{U}) \cap \mathcal{T}$. Let $\mathcal{R}^\vee(U) := \mathcal{T} \cap J^+(J^-(U \cap \Sigma_T) \cap \Sigma_0)$ be defined for any $U \Subset \mathcal{T}$. Note that $\mathcal{R}^\vee := \mathcal{R}^\vee(\mathcal{V}) \Subset \mathcal{R}^\vee(\mathcal{V}')$. First we will show that it is enough to find a Ψ such that

$$(\Phi \,|\, \mathfrak{f})_{\mathcal{R}^\vee(\mathcal{V}')} = (\mathfrak{S}^\dagger\Phi \,|\, \Psi)_{\mathcal{R}^\vee(\mathcal{V}')} \tag{3.12}$$

for all $\Phi \in \Gamma_c(SW)$ where $W := \mathrm{interior}(\mathcal{R}^\vee(\mathcal{V}')) \cup (\mathcal{R}^\vee(\mathcal{V}') \cap \partial\mathcal{T})$ satisfying $M^\dagger\Phi|_{\partial\mathcal{M}} = 0$ and $\Phi|_{\Sigma_T} = 0$: Then, by Definition 3.7 Ψ is a weak solution of (2.4) both restricted to W and restricted to $\hat{W} := \mathrm{interior}(\mathcal{R}^\vee) \cup (\mathcal{R}^\vee \cap \partial\mathcal{T})$. Hence, by Lemma 3.13 and Corollary 3.18 Ψ is a smooth solution on W. Using Proposition 3.4 we obtain that $\Psi|_{\hat{W}\setminus W} = 0$. Hence, we can extend Ψ by zero outside $\hat{W}$ to obtain a weak solution on all of $\mathcal{T}$.

Hence, it remains to find a weak solution on $\mathcal{R}^\vee(\mathcal{V}')$: By Lemma 3.3 and using arguments similar to Proposition 3.5, we notice that the kernel of the operator $\mathfrak{S}_\lambda^\dagger$ acting on

$$\mathrm{dom}\,\mathfrak{S}_\lambda^\dagger := \{\Phi \in \Gamma_c(S\mathcal{T}) \mid \Phi|_{\Sigma_T} = 0, \mathsf{M}^\dagger\Phi|_{\partial\mathcal{M}} = 0\}$$

is trivial. Let now $\ell \colon \mathfrak{S}_\lambda^\dagger(\mathrm{dom}\,\mathfrak{S}_\lambda^\dagger) \to \mathbb{C}$ be the linear functional defined by

$$\ell(\Theta) = (\Phi \,|\, \mathfrak{f})_{\mathcal{R}^\vee(\mathcal{V}')}$$

where Φ satisfies $\mathfrak{S}_\lambda^\dagger\Phi = \Theta$. By the energy inequality (3.5), ℓ is bounded:

$$\ell(\Theta) = (\Phi \,|\, \mathfrak{f})_{\mathcal{R}^\vee(\mathcal{V}')} \leq \|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee(\mathcal{V}'))} \|\Phi\|_{L^2(\mathcal{R}^\vee(\mathcal{V}'))}$$

$$\leq c\|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee(\mathcal{V}'))}\|\mathfrak{S}_\lambda^\dagger\Phi\|_{L^2(\mathcal{R}^\vee(\mathcal{V}'))} = c\|\mathfrak{f}\|_{L^2(\mathcal{R}^\vee(\mathcal{V}'))}\|\Theta\|_{L^2(\mathcal{R}^\vee(\mathcal{V}'))},$$

where in the first inequality we used Cauchy-Schwartz inequality. Then ℓ can be extended to a continuous functional defined on the L^2-completion of $\mathfrak{S}_\lambda^\dagger(\operatorname{dom}\mathfrak{S}_\lambda^\dagger)$ denoted by $\mathscr{H} \subset \mathcal{H}$. Finally, by Riesz representation theorem, there exists a unique element $\Psi \in \mathcal{H}$ such that

$$\ell(\Theta) = (\Theta \mid \Psi)_{\mathcal{R}^\vee(\mathcal{V}')}.$$

for all $\Theta \in \mathfrak{S}_\lambda^\dagger(\operatorname{dom}\mathfrak{S}_\lambda^\dagger)$. Thus, we obtain

$$(\Phi \mid \mathfrak{f})_{\mathcal{R}^\vee(\mathcal{V}')} = \ell(\Theta) = (\Theta \mid \Psi)_{\mathcal{R}^\vee(\mathcal{V}')} = (\mathfrak{S}_\lambda^\dagger \Phi \mid \Psi)_{\mathcal{R}^\vee(\mathcal{V}')}$$

for all $\Phi \in \operatorname{dom}\mathfrak{S}_\lambda^\dagger$. This concludes our proof.   □

4. Global well-posedness of Cauchy problem

Up to now we obtained a weak solution in any time strip $t^{-1}([0,T])$ in Theorem 3.20 and showed that it is actually smooth if the initial data are compactly supported in the interior (or more generally fulfill the compatibility condition (3.11), compare Remark 3.17). We can now easily put everything together to obtain global well-posedness of the Cauchy problem (1.1)–the only comment missing is about negative time:

Remark 4.1. When considering (1.1) on $t^{-1}([-T,0])$ for some $T > 0$ one can map this to the problem in forward time by a time reversal: If $t \mapsto -t$, $\partial_t \mapsto -\partial_t$ and $e_0 \mapsto -e_0$. In particular $-dt$ is then future-directing which makes $\sigma_{\mathfrak{S}_\lambda}(-dt)$ still positive definite. Thus, going again the way via $\mathfrak{S}_\lambda$ we obtain a smooth unique solution of the original Dirac problem (1.1) on all $t^{-1}([-T,0])$ for all $T > 0$.

Proof of Theorem 1.1. Fix $h \in \Gamma_{cc}(S\Sigma_0)$. On account of Theorem 3.20, for any $T \in [0,\infty)$ there exists a weak solution Ψ_T to the Cauchy problem (2.4) in the time strip $\mathcal{T}_T := t^{-1}([0,T])$. Combining Corollary 3.12 with Corollary 3.18, we get in particular that Ψ_T is smooth in the time strip $\mathcal{T}_T$. By applying Lemma 2.6, we finally obtain a smooth solution ψ_T to the Dirac Cauchy problem (1.1) in the time strip $\mathcal{T}_T$. By uniqueness of the solution, see Proposition 3.5, we have $\Psi_{T_1}|_{t^{-1}[0,T_1]} = \Psi_{T_2}|_{t^{-1}[0,T_1]}$ for all $T_1 < T_2$. Hence, we obtain a smooth solution Ψ of (1.1) on $t^{-1}([0,\infty))$. By Remark 4.1 we get analogously a smooth solution $\widetilde{\Psi}$ on $t^{-1}((-\infty,0])$.

It is left to show that ψ and $\widetilde{\Psi}$ glue smoothly at $t = 0$. For that let ϕ be the solution of $\mathsf{D}\phi = f$ on $t^{-1}([-1,1])$, $\phi|_{\Sigma_{-1}=t^{-1}(-1)} = \widetilde{\Psi}|_{\Sigma_{-1}}$ and $M\phi = 0$ on $\partial\mathcal{M}\cap t^{-1}([-1,1])$. By uniqueness we have $\phi|_{t^{-1}([-1,0])} = \widetilde{\Psi}|_{t^{-1}([-1,0])}$ and, hence, $\phi|_{\Sigma_0} = \widetilde{\Psi}_{\Sigma_0} = \Psi_{\Sigma_0}$. Again by uniqueness $\phi|_{t^{-1}([0,1])} = \Psi|_{t^{-1}([0,1])}$ and, thus, by smoothness of ϕ, the solution Ψ and $\widetilde{\Psi}$ glue to a smooth solution of (1.1) on all of $\mathcal{M}$.

For the continuous dependency on the initial data, see below.   □

We are now in the position to discuss the stability of the Cauchy problem. Since the proof is independent on the presence of the boundary and it does rely mostly on functional analytic techniques, we shall omit it and we refer to [3, Section 5] for further details.

Proposition 4.2. *Consider a globally hyperbolic spacetime $\mathcal{M}$ with boundary $\partial\mathcal{M}$ and denote with $S\mathcal{M}$ the spinor bundle over $\mathcal{M}$. Moreover, let $\Gamma_0(S\Sigma_0) \times \Gamma_0(S\mathcal{M})$ denote the space of data satisfying the compatibility condition (3.11). Then the map*

$$\Gamma_0(S\Sigma_0) \times \Gamma_0(S\mathcal{M}) \to \Gamma_{sc}(S\mathcal{M})$$

which assign to $(\mathfrak{h}, \mathfrak{f})$ *a solution* Ψ *to the Cauchy problem* (2.4) *is continuous.*

A byproduct of the well-posedness of the Cauchy problem is the existence of Green operators:

Proposition 4.3. *The Dirac operator is Green hyperbolic, i.e., there exist linear maps, called* advanced/retarded Green operator, $\mathsf{G}^{\pm} \colon \Gamma_{cc}(S\mathcal{M}) \to \Gamma_{sc}(S\mathcal{M})$ *satisfying*

(i) $\mathsf{G}^{\pm} \circ \mathsf{D} f = \mathsf{D} \circ \mathsf{G}^{\pm} f = f$ *for all* $f \in \Gamma_{cc}(S\mathcal{M})$*;*
(ii) $\mathrm{supp}\,(\mathsf{G}^{\pm} f) \subset J^{\pm}(\mathrm{supp}\, f)$ *for all* $f \in \Gamma_{cc}(S\mathcal{M})$*,*

where $J^{\pm}$ *denote the causal future (+) and past (-).*

Proof. Let $f \in \Gamma_{cc}(S\mathcal{M})$ and choose $t_0 \in \mathbb{R}$ such that $\mathrm{supp}\, f \subset J^{+}(\Sigma_{t_0})$. By Theorem 1.1, there exists a unique solution $\psi(f)$ to the Cauchy problem

$$\begin{cases} \mathsf{D}\psi = f \\ \psi|_{\Sigma_0} = 0 \\ \mathsf{M}\Psi|_{\partial\mathcal{M}} = 0. \end{cases}$$

For $f \in \Gamma_{cc}(S\mathcal{M})$ we set $\mathsf{G}^{+} f := \psi$ and notice that $\mathsf{D} \circ \mathsf{G}^{+} f = \mathsf{D}\psi = f$. Note that by the finite speed of propagation, cf. Proposition 3.4, $\mathsf{G}^{+} f \in \Gamma_{sc}(S\mathcal{M})$. Moreover, $\mathsf{G}^{+} \circ \mathsf{D}\psi = \mathsf{G}^{+} f = \psi$ which finishes the proof of (i). By Proposition 3.4, we obtain $\mathrm{supp}\, \mathsf{G}^{+} f \subset J^{+}(f)$ and this conclude the proof of (ii).

The existence of the retarded Green operator G^{-} is proven analogously. $\qquad\square$

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N. Grosse, Mathematisches Institut, Universität Freiburg, 79104 Freiburg, Germany
Email address: `nadine.grosse@math.uni-freiburg.de`

S. Murro, Mathematisches Institut, Universität Freiburg, 79104 Freiburg, Germany
Email address: `simone.murro@math.uni-freiburg.de`