Cohomology in the service of AQFT

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Cohomology

Cohomology is an important part of mathematics and so ubiquitous as to form part of essentially any mathematical theory. It comes in many varieties but there are also unifying aspects.

An Example

- Locally trivial G-bundles. $F \to B$
- Exists open covering $\mathcal{O}_i$ of B and isomorphisms $\phi_i : F \restriction \mathcal{O}_i \to \mathcal{O}_i \times G$.
- $\phi_i \phi_j^{-1}(x, g) = (x, z_{ij} g), \quad x \in \mathcal{O}_i \cap \mathcal{O}_j$.
- $z_{ij} z_{jk} = z_{ik}, \quad \mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$, 1-cocycle.
- Given z exists F_z. $\zeta \mapsto F_z$ not a 1-1 correspondence. Need equivalence relations.
- $z \sim z'$ if there is a $y_i \in G$ such that $z_{ij} y_j = y_i z_{ij}$ if $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$. Equivalence of G-bundles.

Typical Features

- Cohomology has a degree $\in \mathbb{Z}$.
- In low degree cohomology classifies something: $H^1(B, G)$ classifies locally trivial G–bundles.
- Cohomology is the cohomology of some type of mathematical object, here the Čech ohomology of a topological space B.
- A cohomology has coefficients, here the group G.
- May provide examples,
- May help with general theory
- Cohomology classes can often be computed by cohomological methods.
- But cohomology may prove to be just an alternative language. For example the problem of the existence of a field algebra and a gauge group boils down to asking whether the 6-j symbols of the relevant tensor category which form a 3-cocycle are actually a 3-coboundary. There are then 3-j symbols which can be used to embed the category in the category of Hilbert spaces.

Superselection Theory

- Selection Criterion $\pi \restriction \mathcal{O}^\perp \simeq \pi_0 \restriction \mathcal{O}^\perp$, $\mathcal{O} \in \mathcal{K}$.
- Exists unitary $V = V_\mathcal{O}$ with $V\pi(A) = \pi_0(A)V, \quad A \in \mathfrak{A}(O^\perp)$.
  Identify $\pi_0(A)$ with A
- Set $\rho(A) := V\pi(A)V^*$. ρ endomorphism localized in $\mathcal{O}$.
- Things might have gone differently.

Simplices of a Partially Ordered Set

- A 0-simplex of $\mathcal{K}$ is an element $a \in \mathcal{K}$. A 1–simplex b of $\mathcal{K}$ consists of three elements $\partial_0 b, \partial_1 b \subset |b|$.
- A 2-simplex c of $\mathcal{K}$ consists of three 1–simplices $\partial_0 c$, $\partial_1 c$ and $\partial_2 c$ with $\partial_0\partial_0 c = \partial_0\partial_1 c$, $\partial_1\partial_0 c = \partial_0\partial_2 c$ and $\partial_1\partial_1 c = \partial_1\partial_2 c$ together with a further element $|c|$, the support of c, such that $\partial_i\partial_j c \subset |c|$, for all i, j.
- Pick V_a as above and set $z(b) := V_{\partial_0 b} V^*_{\partial_1 b}$ then $z(\partial_1 c) = z(\partial_0) z(\partial_2 c)$ so that z is a 1–cocycle.
- It follows from duality that $z(b) \in \mathfrak{A}(|b|)$. We have local coefficients as in sheaf cohomology. Net cohomology.
- Exist localized endomorphisms $y(a)$ with $z(b) \in (y(\partial_1 b), y(\partial_0 b))$.

Solitons 1972

- 2-spacetime dimensions $\mathfrak{A} \subset \mathfrak{F}$
- $z \in Z^1(\mathfrak{A}) \subset Z^1(\mathfrak{F})$.
- Exist two sets of localized endomorphisms $y_\ell(a)$ and $y_r(a)$ with $z(b) \in (y_\ell(\partial_1 b), y_\ell(\partial_0 b))$ and $z(b) \in (y_r(\partial_1 b), y_r(\partial_0 b))$.
- α-induction

Completeness of sectors 1980

- Free field with gauge group G. Easy to see that there is a sector corresponding to each irreducible representation of G.
- Question of whether there are other sectors remained open for quite some time.
- $$a) \cap_{\partial b' = \partial b} \mathfrak{F}(\mathcal{O} + |b'|) = \mathfrak{F}(\mathcal{O} + \partial_0 b) \vee \mathfrak{F}(\mathcal{O} + \partial_1 b), \quad b \in \Sigma_1,$$

  $$b) \ If (\mathcal{O} + \partial_0 b) \perp (\mathcal{O} + \partial_1 b) \text{ then } \mathfrak{F}(\mathcal{O} + \partial_0 \mathrm{b}) \vee \mathfrak{F}(\mathcal{O} + \partial_1 \mathrm{b})$$

  is canonically isomorphic to $\mathfrak{F}(\mathcal{O} + \partial_0 b) \otimes \mathfrak{F}(\mathcal{O} + \partial_1 b)$, $b \in \Sigma_1$. Abstract conditions that can be verified in the case of the free field.
- Ciolli completeness for the Streater and Wilde model.

Essential Duality

- duality $\mathfrak{A}^d = \mathfrak{A}$, where

$$\mathfrak{A}^d(\mathcal{O}) = \cap\{\mathfrak{A}(\mathcal{O}_1)' : \mathcal{O}_1 \perp \mathcal{O}\}.$$

- essential duality $\mathfrak{A}^{dd} = \mathfrak{A}^d$.
- $Z^1(\mathfrak{A}) \simeq Z^1(\mathfrak{A}^{dd})$.
- Wedge duality implies essential duality.
- The set of representations satisfying essential duality is closed under direct sums and subrepresentations.
- In the absence of duality a representation satisfying the selection criterion, i.e. an object of $\mathrm{Rep}^{\perp}\mathfrak{A}$, yields a cocycle in $Z^1_t(\mathfrak{A}^d)$, the path-independent cocycles in $Z^1(\mathfrak{A}^d)$.

Curved Spacetime

- The advent of curved spacetime revitalized superselection theory. The obvious question being: how does the topology and causal structure of spacetime affect the superselection structure?
- Guido, Longo, J.E.R, Verch (2001)
- $\Sigma_1^\perp = \{b \in \Sigma_1 : \partial_0 b \perp \partial_1 b\}$ has same number of connected components as in Minkowski space. No new solitonic phenomena.
- Theory of sectors goes through if set $\mathcal{K}$ of regular diamonds is directed. Standard use of cohomology. Interesting part of problem left open.

Homotopy

- Two notions of path, usual topological one and one starting from $\mathcal{K}$, where a path is a concatenation of 1–simplices. We suppose that $\mathcal{K}$ is path-connected.
- Both notions of path lead to a notion of homotopy group. If $\mathcal{K}$ is directed, Σ_* admits a contracting homotopy.
- Let $\mathcal{M}$ be arcwise connected and Hausdorff and $\mathcal{K}$ a base for the topology of $\mathcal{M}$ consisting of arcwise and simply connected subsets of $\mathcal{M}$, then $\pi_1(\mathcal{M}) = \pi_1(\mathcal{K})$. (Ruzzi)
- $z \in Z^1(\mathfrak{A})$ and $p \sim q$ then $z(p) = z(q)$. Set

  $$\eta_z([p]) := z(p), \quad [p] \in \pi_1(\mathcal{K}, a_0).$$

  Map from 1–cocycles equivalent in $\mathcal{B}(\mathcal{H}_0)$ to equivalent unitary representations of the homotopy group.

Diamonds

- The approach to superselection sectors in Guido et al was based on the notion of regular diamond. These have the disadvantage that their causal complements may not be pathwise connected.
- Ruzzi improved matters by taking $\mathcal{K}$ to be the set of diamonds.
- Given a spacelike Cauchy surface $\mathcal{C}$ we let $\mathcal{G}(\mathcal{C})$ denote the set of open subsets G of $\mathcal{C}$ of the form $\phi(B)$ for a chart (U, ϕ) of $\mathcal{C}$ where B is an open ball of $\mathbb{R}^3$ with $\mathrm{cl}(B) \subset \phi^{-1}(U)$. A diamond of $\mathcal{M}$ is then a subset $\mathcal{O} = D(G)$ where $G \in \mathcal{G}(\mathcal{C})$ for some spacelike Cauchy surface $\mathcal{C}$. $D(G)$ is the domain of dependence of G.
- $\mathcal{K}$ is a base for the topology of $\mathcal{M}$. A diamond is an open, relatively compact, arcwise and simply connected subset. $D(G)$ is a globally hyperbolic spacetime with spacelike Cauchy surface G.
- The causal complement of a diamond

$$\mathcal{O}^\perp := \{\mathcal{O}_1 \in \mathcal{K} : \mathcal{O}_1 \perp \mathcal{O}\}$$

is pathwise connected in $\mathcal{K}$.

Causal Punctures

- Typically, $\mathcal{K}$ is not directed when Cauchy surfaces of $\mathcal{M}$ are compact. Cannot transport charge to or from infinity. Remove a point.
- The causal puncture of $\mathcal{K}$ at a point $x \in \mathcal{M}$ is

$$\mathcal{K}_x := \{\mathcal{O} \in \mathcal{K} : (\mathcal{O}^-) \perp x\}.$$

- Can also think in terms of a subset of $\mathcal{M}$ $\mathcal{M}_x = \mathcal{M} \backslash X_x = D(\mathcal{C} \backslash \{x\})$ for some spacelike Cauchy surface $\mathcal{C}$ containing $\{x\}$.
- Considered as a spacetime, $\mathcal{M}_x$ is globally hyperbolic but an element $\mathcal{O} \in \mathcal{K}_x$ need not be a diamond of $\mathcal{M}_x$. Still $\mathcal{K}_x$ is a basis for the topology of $\mathcal{M}_x$ and, $\mathcal{M}_x$ being arcwise connected, $\mathcal{K}_x$ is pathwise connected.

Strategy

- First discuss superselection sectors for $\mathcal{K}_x$ for all x and then 'glue' the results together to describe the superselection theory for $\mathcal{K}$. The advantage of studying $\mathcal{K}_x$ is that it behaves in much the same way as Minkowski space.
- Let $\mathfrak{A}_x$ denote the restriction of $\mathfrak{A}$ to $\mathcal{K}_x$. Each $\mathfrak{A}_x$ must satsfy duality.
- If $z \in Z^1(\mathfrak{A})$ is path-independent on $\mathcal{K}_x$ for each $x \in \mathcal{M}$, then z is path-independent on $\mathcal{K}$.
- Hence the 1–cocycles of $z \in Z^1(\mathfrak{A})$ are trivial in $\mathcal{B}(\mathcal{H}_0)$ for an arbitrary 4-dimensional globally hyperbolic spacetime.
- A set of cocycles, $z_x \in Z_t^1(\mathfrak{A}_x)$, $x \in \mathcal{M}$, extends to $Z_t^1(\mathfrak{A})$ if and only if

  $$z_{x_1}(b) = z_{x_2}(b)$$

  whenever $|b| \in \mathcal{K}_{x_1} \cap \mathcal{K}_{x_2}$. A similar result holds for arrows.

Endomorphisms

- Superselection theory comes alive when endomorphisms are introduced.
- $\mathcal{K}_x$ is not necessarily directed and the definition of $y^z(a)$ is a variant on the traditional one.
- Given $z \in Z^1_t(\mathfrak{A}_x)$, $a \in \Sigma_0(\mathcal{K}_x)$ and define

$$y^z_{\mathcal{O}}(a)(A) := z(p)Az(p)^*, \quad A \in \mathfrak{A}(\mathcal{O}_1),\ \mathcal{O}_1 \perp \mathcal{O},$$

  where $x \in \mathcal{O} \in \mathcal{K}$, p is a path in $\mathcal{K}_x$ with $\partial_1 p \subset \mathcal{O}$ and $\partial_0 p = a$. This definition does not depend on the chosen path and, letting $\mathcal{O}$ shrink to $\{x\}$, extends to an endomorphism of $\mathcal{A}^{\perp}(x)$, the C^*–algebra generated by the $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \in \mathcal{K}_x$.
- $z(p)y^z(\partial_1 p)(A) = y^z(\partial_0 p)(A)z(p)$.
- $y^z(a)(\mathfrak{A}(a_1)) \subset \mathfrak{A}(a_1)$ for $a_1 \in \mathcal{K}_x$ with $a \subset a_1$.

Tensor Product

- Tensor product on $Z_t^1(\mathfrak{A}_x)$:
$$(z \otimes z_1)(b) := z(b) y^z(\partial_1 b)(z_1(b)), \quad b \in \Sigma_1(\mathcal{K}_x),$$
- $$(t \otimes s)_a := t_a y^z(a)(s_a), \quad a \in \Sigma_0(\mathcal{K}_x),$$
where $t \in (z, z_1)$, $s \in (z_2, z_3)$.
- The composition law $\otimes$ makes $Z_t^1(\mathfrak{A}_x)$ into a tensor C^*–category.

Conjugates

- Let z be a simple object of $Z_t^1(\mathfrak{A}_x)$ then a conjugate $\overline{z}$ of z is given by
$$\overline{z}(b) := y^{z-1}(\partial_0 b)(z(b)^*), \quad b \in \Sigma_1(\mathcal{K}_x)$$
.
- In a symmetric tensor C^*–category with $(\iota, \iota) = \mathbb{C}$ where every simple object has a conjugate every object with finite statistics has a conjugate.
- Let $\mathcal{T}$ be a symmetric tensor C^*–category with conjugates, subobjects and direct sums, each object having a statistical phase 1 then $\mathcal{T}$ is isomorphic to the symmetric tensor C^*–category of finite dimensional unitary representations of a compact group unique up to isomorphism.

Global Theory

- Results for $Z_t^1(\mathfrak{A}_x)$ for all x can be glued together to give the corresponding results for $Z_t^1(\mathfrak{A})$.
- Given an object z of $Z_t^1(\mathfrak{A})$, let $y_x^z(a)$ denote the endomorphism of $\mathcal{A}^\perp(x)$ associated with the restriction of z to $Z_t^1(\mathfrak{A}_x)$.
- $$y_{x_1}^z(a) \upharpoonright \mathfrak{A}(\mathcal{O}) = y_{x_2}^z(a) \upharpoonright \mathfrak{A}(\mathcal{O}),$$

  whenever $\mathcal{O} \in \mathcal{K}_{x_1} \cap \mathcal{K}_{x_2}$
- If p is a path in $\mathcal{K}_{x_1} \cap \mathcal{K}_{x_2}$ then $y_{x_1}^z(a)(z(p)) = y_{x_2}^z(a)(z(p))$.
- There is a unique symmetry ε for $Z_t^1(\mathfrak{A})$ such that $\varepsilon(z, z_1)_a = \varepsilon_x(z, z_1)$ for $x \perp a$.
- Objects of $Z_t^1(\mathfrak{A})$ with finite statistics have conjugates.
- The restriction tensor *–functor F_x from $Z_t^1(\mathfrak{A})$ to $Z_t^1(\mathfrak{A}_x)$ is full and faithful.

Further Work

- Last part of talk has been based on
G. Ruzzi. Homotopy of posets, net-cohomology, and theory of superselection sectors in global ly hyperbolic spacetimes. Rev. Math. Phys. 17, no.9, (2005), 1021-1070.
There has been further work by Brunetti and Ruzzi on superselection theory in locally covariant quantum field theory. This, too, makes use of cohomology.
- I think we may conclude, that in the course of the years, cohomology has turned into the preferred tool for tackling problems in superselection theory. What is the reason?
- The alternative to using cohomology is to use endomorphisms. Endomorphisms work well when $\mathcal{K}$ is directed and we get endomorphisms of $\mathcal{A}(\mathcal{M})$. In the case of $\mathcal{K}_x$ we got endomorphisms of $\mathcal{A}^\perp(x)$. But in general an endomorphism will need a domain of definition.
- Endomorphisms are used to define the tensor product structure. But this can be defined instead using cocycles:

$$z \otimes z_1(b) = z(b)z(p)z_1(b)z(p)^* \quad \partial_0 p = \partial_1 b,\ \partial_1 p \perp |b|.$$

There is a similar formula for the tensor product of arrows in $Z^1(\mathfrak{A})$. Here we do not need $\mathcal{K}$ to be directed but just connected.