Algebras of observables: a gauge theory example

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The usual kinematics of a gauge theory is:

- fields are connections ∇ for a G-bundle $P \to X$
- these should be identified if related by a gauge transformation

Issue: The quotient space is usually ugly.

One solution: Work with the *stack*, which we denote $Conn/Gauge$.

(Won't define today, but I promise you'll see them all over the place if you learn the definition.)

"The stack of Yang-Mills fields on Lorentzian manifolds" by Benini-Schenkel-Schreiber might be well-suited to this audience.

The dynamics depends upon an action functional

$$S : Conn/Gauge \to \mathbb{R}.$$

We want to study its critical locus: $\{\nabla \text{ such that } dS|_\nabla = 0\}$.

This is a fiber product:

$$\begin{array}{ccc} Crit(S) & \longrightarrow & Conn/Gauge \\ \downarrow & & \downarrow \text{zero section} \\ Conn/Gauge & \xrightarrow{dS} & T^*Conn/Gauge \end{array}$$

Nowadays, some people suggest you take the *derived* fiber product, which is a *derived stack* known as the derived critical locus $dCrit(S)$.

Issue: Quite abstract, and there is no quantization prescription.

Let's be practical and think about perturbation theory around $\nabla \in Crit(dS)$. For simplicity, suppose the bundle $P \to X$ is trivial.

Let's eyeball the computation of the tangent space.

For the kinematics, we should have the linearization of the gauge action:

$$\underline{-1} \qquad\qquad\qquad \underline{0}$$

$$T_\nabla Gauge \xrightarrow{d(act)_\nabla} T_\nabla Conn$$

Cokernel is obvious linearization, and kernel is linearized stabilizer.

These are nice spaces:

$$\begin{array}{ccc} \underline{-1} & & \underline{0} \\ \text{Lie}(Gauge) & \xrightarrow{d(act)_\nabla} & T_\nabla Conn \\ \| & & \| \\ \Omega^0(X) \otimes \mathfrak{g} & & \Omega^1(X) \otimes \mathfrak{g} \end{array}$$

The derived fiber product is given by the derived kernel of dS (mapping cocone) along the tangent space at ∇:

$$\mathbb{L}Ker(dS_\nabla) = \text{Cocone}(T_\nabla Conn/Gauge \xrightarrow{dS_\nabla} T^*_\nabla Conn/Gauge)$$

Hence the derived fiber product is:

$$\begin{array}{ccccccc} \underline{-1} & & \underline{0} & & \underline{1} & & \underline{2} \\ T_{\nabla}Gauge & \xrightarrow{d(act)_{\nabla}} & T_{\nabla}Conn & \xrightarrow{dS_{\nabla}} & T^*_{\nabla}Conn & \xrightarrow{d(act)^*_{\nabla}} & T^*_{\nabla}Gauge \\ \| & & \| & & \| & & \| \\ \Omega^0(X)\otimes\mathfrak{g} & & \Omega^1(X)\otimes\mathfrak{g} & & \Omega^{d-1}(X)\otimes\mathfrak{g} & & \Omega^d(X)\otimes\mathfrak{g} \end{array}$$

These are precisely the fields you write in BV/BRST formalism!

$$\begin{array}{cccc} \underline{-1} & \underline{0} & \underline{1} & \underline{2} \end{array}$$

$$\Omega^0(X) \otimes \mathfrak{g} \xrightarrow{d(act)_\nabla} \Omega^1(X) \otimes \mathfrak{g} \xrightarrow{dS_\nabla} \Omega^{d-1}(X) \otimes \mathfrak{g} \xrightarrow{d(act)^*_\nabla} \Omega^d(X) \otimes \mathfrak{g}$$

If you work through the whole procedure, you find that the classical BV theory matches the derived deformation theory of $dCrit(S)$.

Caveat: Not yet a theorem because a sufficient theory of ∞-dimensional derived differential geometry has not been developed (so far as I know).

Upside: We do have a prescription for perturbative quantization.

It has been put on a rigorous footing recently, thanks to many people. See the systematic treatment by Klaus & collaborators.

Immediately from the prescription, you obtain a differential graded (=dg) algebra of observables on each region of spacetime. In this sense, you obtain a rather minimal dg generalization of pAQFT.

Here's the untechnical idea from my collaboration with Kasia:

A *dg QFT model* on a spacetime $\mathcal{M}$ is a functor

$$\mathfrak{A} : \mathbf{Caus}(\mathcal{M}) \to \mathbf{Alg}^*(\mathbf{Ch}(\mathrm{TVS}))$$

so that each $\mathfrak{A}(\mathcal{O})$ is a locally convex unital $*$-dg algebra satisfying *Einstein causality*: spacelike-separated observables commute at the level of cohomology.

That is, for $\mathcal{O}_1, \mathcal{O}_2 \in \mathbf{Caus}(\mathcal{M})$ that are spacelike to each other, the commutator $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)]$ is exact in $\mathfrak{A}(\mathcal{O}')$ for any $\mathcal{O}' \in \mathbf{Caus}(\mathcal{M})$ that contains both $\mathcal{O}_1$ and $\mathcal{O}_2$.

We also phrase time-slice axiom as a cohomology-level statement. It's not obvious (to me at least) how to generalize all the usual conditions.

Question: Is there anything interesting here?

Kasia and I recently examined free scalar theories this way and recovered the standard answers, of course. The procedure is overkill here.

The rest of the talk will try to convince you the answer can be "yes," by discussing the example of Chern-Simons theory.

Kasia and I hope to explore other examples, like Yang-Mills theories, and welcome others' help!

Issue: A stack is not (usually) encoded by its algebra of functions.

The deformation theory of a point on a stack *is* algebraic, however, and the BV/BRST formalism exploits this fact. In a sense, this is why this naive dg generalization of pAQFT appears.

Question: Is there a generalization of this dg version of AQFT that would apply to global stacks?

A useful constraint is that such a global quantum definition ought to recover the perturbative prescription when you work around a fixed solution.

Question: How do triumphs of AQFT generalize in this naive dg setting? For example, how does DHR theory change?

Speculation: The dependence on dimension changes. I suggest you get an E_n-monoidal ∞-category for n-dimensional theories, where E_n means the operad of little n-disks. For ordinary categories

- E_1 corresponds to monoidal,
- E_2 corresponds to braided monoidal,
- $E_{\geq 3}$ corresponds to symmetric monoidal,

but for higher categories the E_n are all different.

X – oriented 3-dimensional smooth manifold

G – compact Lie group with nondegenerate pairing $\langle -, - \rangle$ on $\mathfrak{g}$ (e.g., Killing form)

Chern-Simons action:

$$CS(d+A) = \frac{1}{2}\int_X \langle A \wedge dA \rangle + \frac{1}{3!}\int_X \langle A \wedge [A, A] \rangle$$

Equation of motion: $F_A = 0$ (zero curvature or Maurer-Cartan equation)

Note: No dependence on signature.

Mathematical appeal: Classical CS theory studies G-local systems on X, a topic beloved by topologists and representation theorists. Its quantization has produced further intriguing mathematics.

Physical appeal: It is the TFT *par excellence*, and a great toy example. Moreover, abelian CS plays a key role in the effective field theory of the quantum Hall effect.

The perturbative quantization was explored by many physicists (see, e.g., Guadagnini-Martellini-Mintchev). Axelrod-Singer and Kontsevich (unpublished) constructed mathematically its BV quantization in the early 1990s.

Up to equivalence, every perturbative quantization is determined by an $\hbar$-dependent level $\lambda \in \hbar H^3_{\mathrm{Lie}}(\mathfrak{g})[[\hbar]]$.

Everything is encoded in the dg Lie algebra $\Omega^*(X) \otimes \mathfrak{g}$.

Lie bracket: $[\alpha \otimes x, \beta \otimes y] = \alpha \wedge \beta \otimes [x, y]$

differential: $d(\alpha \otimes x) = (d\alpha) \otimes x$

Every dg Lie algebra has an associated Maurer-Cartan equation, and this one recovers usual EoM of Chern-Simons theory.

If you unpack the BV formalism in this case, the dg commutative algebra of classical observables is

$$\mathrm{Obs}^{\mathrm{cl}}(X) = \mathrm{C}^*_{\mathrm{Lie}}(\Omega^*(X) \otimes \mathfrak{g}).$$

De Rham cohomology is easy to compute, so we can quickly analyze the classical observables.

The functoriality of the de Rham complex ensures we have a functor

$$\mathrm{Obs}^{\mathrm{cl}} : \mathrm{Mfld}_3^{or} \to \mathbf{Ch}$$

Note the similarity with covariance paradigm.

$\underline{X = \mathbb{R}^3}$:

$\Omega^*(\mathbb{R}^3) \otimes \mathfrak{g} \simeq \mathfrak{g}$ by Poincaré lemma so

$$\mathrm{Obs}^{\mathrm{cl}}(\mathbb{R}^3) \simeq \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$$

We just have functions on *ghosts*, as only "gauge symmetry" is relevant very locally. (This is a shadow of the stack structure.)

This might seem weird and unphysical, but it's a familiar feature of cohomology, which is boring locally *by design*.

To get something interesting, we need X to have interesting topology. But we already know that the important observables live on a circle: the Wilson loops!

$\underline{X = S^1 \times \mathbb{R}^2}$:

$\Omega^*(S^1) \simeq \mathbb{C}[\epsilon]$ with $|\epsilon| = 1$

Hence by Künneth and Poincaré, we see

$$\mathrm{Obs}^{\mathrm{cl}}(S^1 \times \mathbb{R}^2) \simeq \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}[\epsilon]) = \mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g}, \widehat{\mathrm{Sym}}(\mathfrak{g}^*))$$

Thus

$$H^0\mathrm{Obs}^{\mathrm{cl}} = \widehat{\mathrm{Sym}}(\mathfrak{g}^*)^{\mathfrak{g}}$$

which are the infinitesimal class functions.

For example, the *character* of a finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ lives here:

$$\mathrm{ch}_\rho(x) = \mathrm{tr}_V(\exp(\rho(x)))$$

On-shell it agrees with a classical Wilson loop, as it is the path-ordered exponential evaluated on a flat connection.

For simplicity, we restrict to abelian CS: $\mathfrak{g} = \mathbb{C}$.

Consider a closed genus g surface Σ_g. The cohomology of the fields

$$H^*(\Sigma_g \times \mathbb{R})[1] = \begin{array}{ccc} \underline{-1} & \underline{0} & \underline{1} \\ \mathbb{C} & \mathbb{C}^{2g} & \mathbb{C} \end{array}$$

has a symplectic structure via Poincaré duality. So $H^*\mathrm{Obs}^{\mathrm{cl}}(\Sigma_g \times \mathbb{R})$ has a Poisson structure.

Note the subalgebra

$$\mathrm{Sym}(H_1(\Sigma_g)) \cong \mathbb{C}[\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g]$$

in degree 0, with $\{\alpha_i, \beta_j\} = \delta_{ij}$ and other brackets trivial.

Claim: The standard BV quantization determines a deformation quantization of $H^*\mathrm{Obs}^{\mathrm{cl}}(\Sigma_g \times \mathbb{R})$. In particular, the subalgebra becomes the Weyl algebra.

Costello and I showed this in our book, in a broader analysis of abelian Chern-Simons theory. You can also use the arguments that Kasia and I develop in our scalar field analysis, i.e., a pAQFT approach. (These constructions should have nice connections with work of Benini–Schenkel–Szabo.)

Note: This algebra is not apparent locally (i.e., on $\mathbb{R}^2 \times \mathbb{R}$) but emerges by the local-to-global nature of cohomology.

The idea behind the claim is to use local constancy:

$$\begin{array}{ccc}
A \otimes B & & \mathrm{Obs}(\Sigma \times (-t,t)) \otimes \mathrm{Obs}(\Sigma \times (-t,t)) \\
& & \downarrow{\scriptstyle id\otimes\tau_T} \\
A \otimes \tau_T(B) & & \mathrm{Obs}(\Sigma \times (-t,t)) \otimes \mathrm{Obs}(\Sigma \times (T-t,T+t)) \\
& & \downarrow \\
A \cdot \tau_T(B) & & \mathrm{Obs}(\Sigma \times (-t,T+t)) \\
& & {\scriptstyle \simeq}\uparrow \\
\text{“}A \star B\text{”} & & \mathrm{Obs}(\Sigma \times (-t,t))
\end{array}$$

This product is only defined up to exact terms, so we get a strict algebra only at the level of cohomology.

Using modern homotopical algebra, you can go quite a bit further.

You might have heard about the little n-disks operad E_n. If so, consider how CS observables behave when you work with open disks inside $\mathbb{R}^3$.

Claim:

- $\mathrm{Obs}^{\mathrm{cl}}$ determines an algebra over the little 3-disks operad.
- A BV quantization deforms this E_3-algebra structure.

Question: Can one extract anything concrete from this?

Theorem: (Costello-Francis-G.)

There is a natural bijection between the following:

- perturbative quantizations of Chern-Simons theory on $\mathbb{R}^3$, up to equivalence, and
- braided monoidal deformations of $\mathrm{Rep}_{fin}(U\mathfrak{g})$ over $\mathbb{C}[[\hbar]]$, up to braided monoidal equivalence.

Key idea 1: There is a filtered Koszul duality of dg algebras between $U\mathfrak{g}$ and $C^*_{\mathrm{Lie}}(\mathfrak{g})$, and this determines an equivalence of symmetric monoidal dg categories

$$\mathrm{Rep}^{dg}_{fin}(U\mathfrak{g}) \simeq \mathrm{Perf}(C^*_{\mathrm{Lie}}(\mathfrak{g})).$$

Hence every braided monoidal deformation of $\mathrm{Perf}(C^*_{\mathrm{Lie}}(\mathfrak{g}))$ determines a braided monoidal deformation of $\mathrm{Rep}(U\mathfrak{g})$.

Such braided monoidal structures are, in essence, *quantum groups*.

Key idea 2: Lurie has shown that the left modules of an E_3 algebra form an E_2-monoidal ∞-category. (This is the higher categorical version of a braided monoidal structure.)

Finally, we saw that $\mathrm{Obs}^{\mathrm{cl}}$ is equivalent to $\mathrm{C}^*_{\mathrm{Lie}}(\mathfrak{g})$ as an E_3-algebra.

Putting everything together, we see that every BV quantization determines a braided monoidal deformation of representations of $\mathfrak{g}$.

To make the bijection concrete, we describe Wilson line defects inside this BV/factorization algebra framework, and then show how perturbative computations encode the R-matrix.

I hope analogs of these arguments can be realized in pAQFT.