Some ideas about constructive tensor field theory

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Outline

Random tensors, random spaces

Loop Vertex Expansion

A constructive result for tensors

Random tensors, random spaces

Random tensors, random spaces
- Why?
- How?

Loop Vertex Expansion

A constructive result for tensors

Why tensor fields?

1. Generalize matrix models to higher dimensions
   - w.r.t. their symmetry properties,
   - provide a theory of random spaces.

2. Define a canonical way of summing over spaces

3. Implement a geometrogenesis scenario
   - spacetime from scratch,
   - background independent.

Invariant actions

Symmetry

Consider $T, \overline{T} : \mathbb{Z}^D \to \mathbb{C}$, complex rank D tensors with *no symmetry*.

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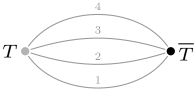
- Invariants as D-coloured graphs



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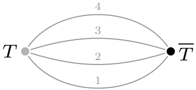

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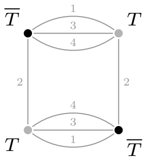

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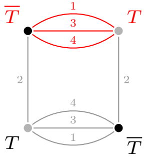

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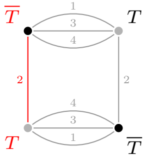

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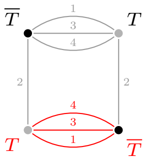

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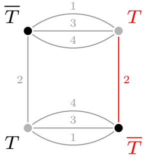

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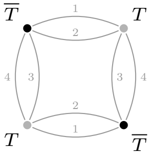

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$$S(T,\overline{T}) = T\cdot\overline{T} + \sum_{\mathcal{B}\in\mathfrak{I}} g_{\mathcal{B}}\,\mathrm{Tr}_{\mathcal{B}}[T,\overline{T}],$$

$$\mathfrak{I}\subset\{D\text{-coloured graphs of order }\geqslant 4\}$$ interaction vertices

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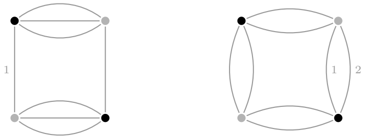

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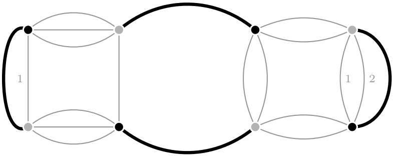

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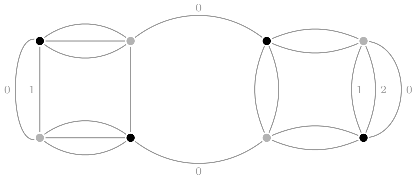

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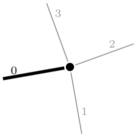

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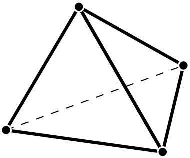

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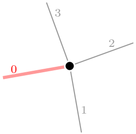
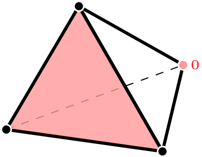


$(D-1)$-face



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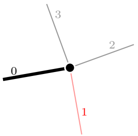

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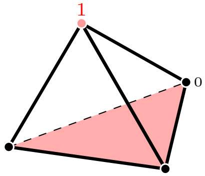

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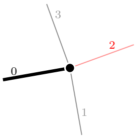

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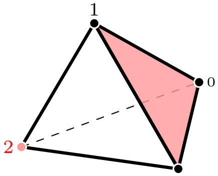

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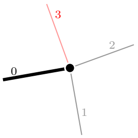

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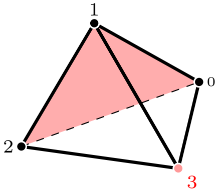

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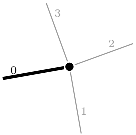

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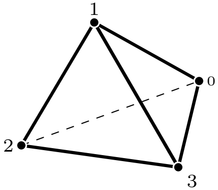

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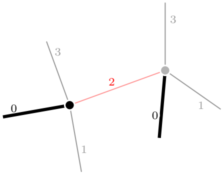
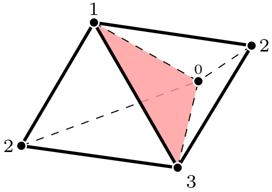


gluing



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- How?

A constructive result for tensors

Constructive field theory

A functional integral point of view

- Aim: get some control on connected quantities via the derivation of tractable formulas for the *logarithm* of correlation functions (say).

Constructive field theory

The classical approach

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- How? By finding an expansion which interpolates between the functional integral and the perturbative series.

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- How? By finding an expansion which interpolates between the functional integral and the perturbative series.
- Tools: With cluster and Mayer expansions which are both a clever application of the forest formula.

The BKAR forest formula

- Fix an integer $n \geqslant 2$.
- f a function of $\frac{n(n-1)}{2}$ variables x_ℓ, sufficiently differentiable.
- K_n, complete graph on $\{1, 2, \ldots, n\}$. $\#E(K_n) = \frac{n(n-1)}{2}$

Then,

$$f(1, 1, \ldots, 1) = \sum_{\mathcal{F}} \int dw_{\mathcal{F}} \, \partial_{\mathcal{F}} f(X^{\mathcal{F}}(w_{\mathcal{F}}))$$

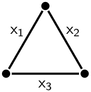
where

- the sum is over spanning forests of K_n,
- $\int dw_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \int_0^1 dw_\ell$,
- $\partial_{\mathcal{F}} := \prod_{\ell \in E(\mathcal{F})} \frac{\partial}{\partial x_\ell}$,
- $X^{\mathcal{F}} = (x_\ell^{\mathcal{F}})_{\ell \in E(K_n)}$ – evaluation point of $\partial_{\mathcal{F}} f$.

The forest formula

An example

$$f(1,1,1) =$$



$=$

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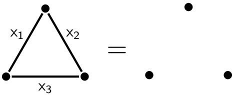
$$f(1,1,1) = f(0,0,0)$$



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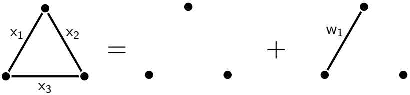

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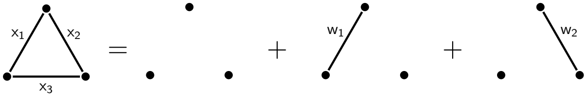

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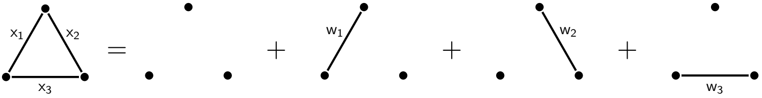

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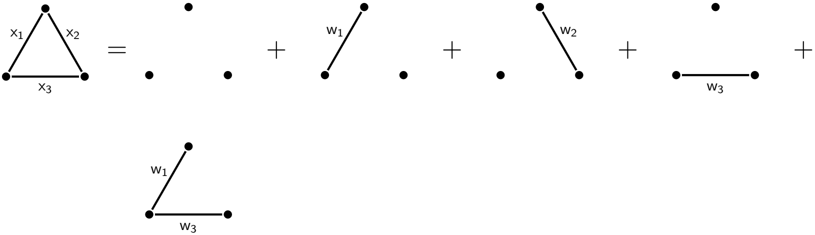
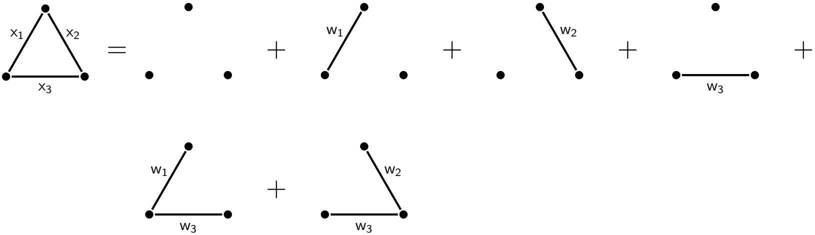


The forest formula

An example

$$f(1,1,1) = f(0,0,0) + \int_0^1 \partial_{x_1} f(w_1,0,0)\, dw_1 + \int_0^1 \partial_{x_2} f(0,w_2,0)\, dw_2 + \int_0^1 \partial_{x_3} f(0,0,w_3)\, dw_3 + \iint \partial^2_{x_1,x_3} f(w_1, \min\{w_1,w_3\}, w_3) + \iint \partial^2_{x_2,x_3} f(\min\{w_2,w_3\}, w_2, w_3)$$



The forest formula

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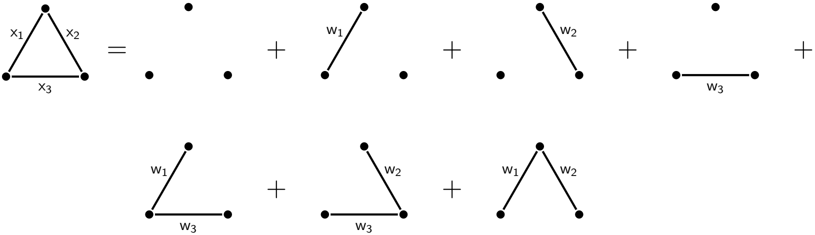
$$
\begin{aligned}
f(1,1,1) = {} & f(0,0,0) + \int_0^1 \partial_{x_1} f(w_1,0,0)\,dw_1 + \int_0^1 \partial_{x_2} f(0,w_2,0)\,dw_2 + \\
& \int_0^1 \partial_{x_3} f(0,0,w_3)\,dw_3 + \iint \partial^2_{x_1,x_3} f(w_1,\min\{w_1,w_3\},w_3) + \\
& \iint \partial^2_{x_2,x_3} f(\min\{w_2,w_3\},w_2,w_3) + \iint \partial^2_{x_1,x_2} f(w_1,w_2,\min\{w_1,w_2\}).
\end{aligned}
$$



Constructive field theory

The classical approach

- Aim: get some control on connected quantities via the derivation of tractable formulas for the *logarithm* of correlation functions (say).
- How? By finding an expansion which interpolates between the functional integral and the perturbative series.
- Tools: With cluster and Mayer expansions which are both a clever application of the forest formula.
- But classical constructive techniques are unsuited to matrices.

Loop Vertex Expansion

Motivations

LVE $=$ main constructive tool for matrices and tensors

- Originally designed for random matrices. [Rivasseau 2007]
- Initial goals:
  1. Constructive ϕ_4^{*4},
  2. Simplify Bosonic constructive theory.

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2. Bosonic perturbative series cannot be resumed as easily as Fermionic ones.

$$S(\lambda) = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} \sum_{G, n(G)=n} A_G$$

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$$= \sum_{n\geqslant 0} \frac{\lambda^n}{n!} \sum_{T, n(T)=n} A_T, \quad A_T = \sum_{G\supset T} w(G,T) A_G$$

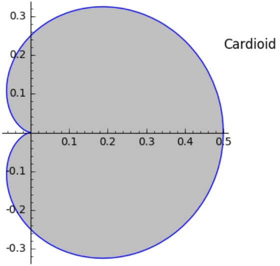
Loop Vertex Expansion

Analyticity of the free energy of ϕ_0^4

$$Z(g) = \int_{\mathbb{R}} e^{-\frac{g}{2}\phi^4} \, d\mu(\phi), \quad d\mu(\phi) = \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$$

Theorem

$\log Z$ *is analytic in the cardioid domain* $\left\{g \in \mathbb{C} : |g| < \frac{1}{2}\cos^2\left(\frac{1}{2}\arg g\right)\right\}$.



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Proof. LVE is made of 2 ingredients:

1. Intermediate field representation,
2. Forest formula.

Loop Vertex Expansion

Analyticity of the free energy of ϕ^4_0

1. Intermediate field representation: $e^{-\frac{g}{2}\phi^4} = \int_{\mathbb{R}} e^{\imath\lambda\sigma\phi^2}\, d\mu(\sigma)$, $\lambda := \sqrt{g}$

$$
\begin{aligned}
Z(g) &= \int_{\mathbb{R}} e^{-\frac{g}{2}\phi^4}\, d\mu(\phi) \\
&= \int_{\mathbb{R}} e^{V(\sigma)}\, d\mu(\sigma), \quad V(\sigma) = -\tfrac{1}{2}\log(1 - \imath\lambda\sigma).
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$$

Loop Vertex Expansion

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$\frac{3}{2}$. Replication of fields:

$$Z(g) = \sum_{n\geqslant 0} \frac{1}{n!} \int_{\mathbb{R}} V(\sigma)^n\, d\mu(\sigma) = \sum_{n\geqslant 0} \frac{1}{n!} \int_{\mathbb{R}^n} \Big(\prod_{i=1}^{n} V(\sigma_i)\Big)\, d\mu_{\mathbb{1}_n}(\vec{\sigma}).$$

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2. Forest formula:

$$Z = \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\mathcal{F}\subset K_n} \int dw_{\mathcal{F}} \int \Big[\prod_{\ell\in E(\mathcal{F})} \frac{\partial}{\partial\sigma_{i(\ell)}} \frac{\partial}{\partial\sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i)\Big]\, d\mu_{C^{\mathcal{F}}(w)}(\vec{\sigma})$$

Loop Vertex Expansion

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$$\log Z = \sum_{n\geqslant 1} \frac{1}{n!} \sum_{\mathcal{T}\subset K_n} \int dw_{\mathcal{T}} \int \Big[\prod_{\ell\in E(\mathcal{T})} \frac{\partial}{\partial\sigma_{i(\ell)}} \frac{\partial}{\partial\sigma_{j(\ell)}} \prod_{i=1}^n V(\sigma_i)\Big]\, d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).$$

Loop Vertex Expansion

Analyticity of the free energy of ϕ^4_0

$$\log Z = \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \Big[\prod_{\ell \in E(\mathcal{T})} \frac{\partial}{\partial \sigma_{i(\ell)}} \frac{\partial}{\partial \sigma_{j(\ell)}} \prod_{i=1}^{n} V(\sigma_i) \Big] \, d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma})$$

$$= \frac{1}{2} \sum_{n \geqslant 1} \frac{(-g/2)^{n-1}}{n!} \sum_{\mathcal{T} \subset K_n} \int dw_{\mathcal{T}} \int \Big[\prod_{i=1}^{n} \frac{(d_i - 1)!}{(1 - \imath \lambda \sigma_i)^{d_i}} \Big] \, d\mu_{C^{\mathcal{T}}(w)}(\vec{\sigma}).$$

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Analyticity of the free energy of ϕ_0^4

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Using $|1-\imath\lambda\sigma| \geqslant \cos(\frac{1}{2}\arg g)$, we get

$$|\log Z| \leqslant \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \Big(\frac{|g|}{2\cos^2(\frac{1}{2}\arg g)} \Big)^{n-1} \sum_{\mathcal{T}\subset K_n} \prod_{i=1}^{n} (d_i-1)!$$

$$\leqslant 2 \sum_{n=1}^{\infty} \Big(\frac{2|g|}{\cos^2(\frac{1}{2}\arg g)} \Big)^{n-1}$$

which is convergent for all $g\in\mathbb{C}$ such that $|g| < \frac{1}{2}\cos^2(\frac{1}{2}\arg g)$. $\square$

A constructive result for tensors

Random tensors, random spaces

Loop Vertex Expansion

A constructive result for tensors

The T_4^4 field theory

- Tensors:

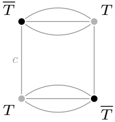
$$T : \mathbb{Z}^4 \to \mathbb{C}, \qquad T_{\boldsymbol{n}}, \overline{T}_{\overline{\boldsymbol{n}}} \text{ with } \boldsymbol{n}, \overline{\boldsymbol{n}} \in \mathbb{Z}^4.$$

- Free action:

$$C_{\boldsymbol{n},\overline{\boldsymbol{n}}} = \delta_{\boldsymbol{n},\overline{\boldsymbol{n}}} \frac{(\mathbf{1}_{\leqslant j_{\max}})_{\boldsymbol{n}\overline{\boldsymbol{n}}}}{\boldsymbol{n}^2 + 1}, \quad \boldsymbol{n}^2 := n_1^2 + n_2^2 + n_3^2 + n_4^2.$$

- Interactions:

$$V(T, \overline{T}) = \tfrac{g}{2} \sum_{c=1}^{4} V_c(T, \overline{T}), \qquad V_c(T, \overline{T}) =$$



Lemma

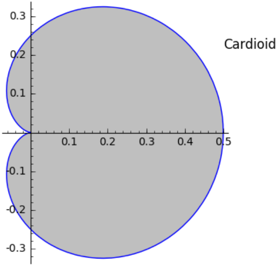
T_4^4 is (super-)renormalizable to all orders of perturbation with a power-counting similar to ϕ_3^4.

The T_4^4 field theory

Analyticity

Theorem (Rivasseau, V.-T. 2017)

There exists $\rho > 0$ such that $\lim_{j_{max} \to \infty} \log Z_{j_{max}}(g)$ is analytic in the cardioid domain defined by $|\arg g| < \pi$ and $|g| < \rho \cos^2(\frac{1}{2} \arg g)$.



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Remarks

- *Intermediate field is a matrix,*
- *Renormalization needed,*
- *Multiscale analysis required,*
- *Forests turn into jungles,*
- *Non-perturbative bounds necessary.*

Conclusion and perspectives

- Regarding T_4^4, one could also prove Borel summability of the connected correlation functions.
- LVE makes Bosons as convergent as Fermions (in dimension 0).

- T_5^4 (just renormalisable, asymptotically free)
- New Loop Vertex Representation [Rivasseau 2017]
- Inductive approach to LVE [Fang-Jie Zhao 2019]
- Simplify Bosonic constructive theory?

Thank you for your attention

The general strategy

0. Renormalised partition function:

$$Z_{j_{\max}}(g) = \mathcal{N} \int e^{-\frac{g}{2}\sum_c V_c(T,\overline{T}) + T\cdot\overline{T}\left(\sum_{G\in\mathcal{M}} \frac{(-g)^{|G|}}{S_G}\,\delta_G\right)}\, d\mu_C(T,\overline{T}).$$

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1. Intermediate field representation: $\sigma_c \in \mathrm{Herm}_{M^{j_{\max}}},\ c = 1,2,3,4$

$$Z_{j_{\max}}(g) = \mathcal{N} e^{\delta_t} \int e^{-\operatorname{Tr}\log(\mathbb{I} - \Sigma) - \imath\lambda \sum_c \delta_m^c \operatorname{Tr}_c \sigma_c}\, d\nu_{\mathbb{I}}(\vec{\sigma})$$

$$\lambda = \sqrt{g}$$

$$\Sigma = \imath\lambda C^{1/2}\sigma C^{1/2}$$

$$\sigma = \sigma_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \mathbb{I}_1 \otimes \sigma_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 + \cdots$$

The general strategy

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2. Renormalised *action*:

$$Z_{j_{\max}}(g) = \int e^{-\operatorname{Tr}\log_3(\mathbb{I}-U) - \operatorname{Tr}(D_1\Sigma^2) - \frac{\lambda^2}{2}:\vec{\sigma}\cdot Q\vec{\sigma}:}\, d\nu_{\mathbb{I}}(\vec{\sigma})$$

$$U = \Sigma + D_1 + D_2$$

$$D_1 = -\lambda^2 C^{1/2} A^{\mathrm{r}}_{\mathcal{M}_1} C^{1/2}, \qquad D_2 = \lambda^4 C^{1/2} A^{\mathrm{r}}_{\mathcal{M}_2} C^{1/2}$$

$$\log_3(\mathbb{I}-U) = \log(\mathbb{I}-U) + U + \frac{U^2}{2}$$

The general strategy

3. Multiscale decomposition:

$$C_{\leqslant j} := \delta_{\boldsymbol{n},\bar{\boldsymbol{n}}} \frac{(\mathbf{1}_{\leqslant j})_{\boldsymbol{n}\bar{\boldsymbol{n}}}}{\boldsymbol{n}^2 + 1}$$

$$V_{\leqslant j} = \mathrm{Tr}\log_3[\mathbb{I} - U_{\leqslant j}] + \mathrm{Tr}[D_{1,\leqslant j}\Sigma^2_{\leqslant j}] + \frac{\lambda^2}{2} :\vec{\sigma}\cdot Q_{\leqslant j}\vec{\sigma}:$$

$$V_{\leqslant j_{\max}} = \sum_{j=1}^{j_{\max}} (V_{\leqslant j} - V_{\leqslant j-1}) =: \sum_{j=1}^{j_{\max}} V_j$$

$$Z_{j_{\max}}(g) = \int \prod_j e^{-V_j}\, d\nu_{\mathbb{I}}(\vec{\sigma}) = \int e^{-\sum_j \bar{\chi}_j W_j(\sigma)\chi_j}\, d\nu_{\mathbb{I}}(\vec{\sigma}) d\mu(\bar{\chi}, \chi)$$

$$W_j = e^{-V_j} - 1$$

The general strategy

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$$Z_{j_{\max}}(g) = \mathcal{N} \int e^{-\frac{g}{2}\sum_c V_c(T,\overline{T}) + T\cdot\overline{T}\left(\sum_{G\in\mathcal{M}} \frac{(-g)^{|G|}}{S_G}\,\delta_G\right)}\, d\mu_C(T,\overline{T}).$$

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The general strategy

4. Multiscale Loop Vertex Expansion: [Gurau, Rivasseau 2014]
    - 2 forest formulas on top of each other
    - First, a Bosonic forest then a Fermionic one

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$$\log Z_{j_{\max}}(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\boldsymbol{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}}\, \partial_{\mathcal{J}} \Big[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \big(-\bar{\chi}^{\mathcal{B}}_{j_a} W_{j_a}(\vec{\sigma}^a) \chi^{\mathcal{B}}_{j_a}\big) \Big]$$

- $\boldsymbol{w}_{\mathcal{J}}$ = weakening parameters w_ℓ, $\ell \in E(\mathcal{J})$
- $\nu_{\mathcal{J}}$ = interpolated Gaussian Bosonic and Fermionic measures
- $\partial_{\mathcal{J}}$ = derivatives with respect to the σ-, χ- and $\bar{\chi}$-fields

The general strategy

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$$\log Z_{j_{\max}}(g) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}\ \text{tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\boldsymbol{w}_{\mathcal{J}} \int d\nu_{\mathcal{J}}\, \partial_{\mathcal{J}} \Big[\prod_{\mathcal{B}} \prod_{a\in\mathcal{B}} (-\bar{\chi}^{\mathcal{B}}_{j_a} W_{j_a}(\vec{\sigma}^a) \chi^{\mathcal{B}}_{j_a}) \Big]$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}\ \text{tree}} \sum_{j_1=1}^{j_{\max}} \cdots \sum_{j_n=1}^{j_{\max}} \int d\boldsymbol{w}_{\mathcal{J}} \prod_{\mathcal{B}} \Big(\prod_{\substack{a,b\in\mathcal{B} \\ a\neq b}} (1-\delta_{j_a j_b}) \Big) I_{\mathcal{B}},$$

$$I_{\mathcal{B}} = \int \partial_{\mathcal{B}} \prod_{a\in\mathcal{B}} W_{j_a}(\vec{\sigma}^a)\, d\nu_{\mathcal{B}} = \sum_{\mathrm{G}} \int \Big(\prod_{a\in\mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a)} \Big) A_{\mathrm{G}}(\vec{\sigma})\, d\nu_{\mathcal{B}}(\vec{\sigma}).$$

Graphs G are plane forests.

The general strategy
Bosonic bounds

$$I_{\mathcal{B}} = \sum_{G} \int \Big(\prod_{a \in \mathcal{B}} e^{-V_{j_a}(\vec{\sigma}^a)} \Big) A_G(\vec{\sigma})\, d\nu_{\mathcal{B}}(\vec{\sigma})$$

$$|I_{\mathcal{B}}| \leqslant \underbrace{\Big(\int \prod_{a \in \mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}} \Big)^{1/2}}_{I_{\mathcal{B}}^{NP},\ \text{non-perturbative}} \sum_{G} \underbrace{\Big(\int |A_G(\vec{\sigma})|^2\, d\nu_{\mathcal{B}} \Big)^{1/2}}_{I_{\mathcal{B}}^{P},\ \text{perturbative}}$$

The general strategy

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5. Non-perturbative bound:

Theorem

For ρ small enough and for any value of the w interpolating parameters,

$$I_{\mathcal{B}}^{NP} = \int \prod_{a\in\mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}} \leqslant O(1)^{|\mathcal{B}|}.$$

The general strategy

Bosonic bounds

$$|I_{\mathcal{B}}| \leqslant \underbrace{\Big(\int \prod_{a\in\mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}}\Big)^{1/2}}_{I_{\mathcal{B}}^{NP},\ \text{non-perturbative}} \sum_{\mathrm{G}} \underbrace{\Big(\int |A_{\mathrm{G}}(\vec{\sigma})|^2\, d\nu_{\mathcal{B}}\Big)^{1/2}}_{I_{\mathcal{B}}^{P},\ \text{perturbative}}$$

6. Perturbative bound:

Theorem

Let $\mathcal{B}$ be a Bosonic block. Then there exists $K \in \mathbb{R}_+^*$ such that

$$I_{\mathcal{B}}^P(\mathrm{G}) \leqslant K^{|\mathcal{B}|-1}\big((|\mathcal{B}|-1)!\big)^{37/2} \rho^{e(\mathrm{G})} \prod_{a\in\mathcal{B}} M^{-\frac{1}{48}j_a}.$$

Non-perturbative bounds

Theorem

For ρ small enough and for any value of the w interpolating parameters,

$$I_{\mathcal{B}}^{NP} = \int \prod_{a\in\mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}} \leqslant O(1)^{|\mathcal{B}|} e^{O(1)\rho^{3/2}|\mathcal{B}|}.$$

Proof.

1. Expand each node:

$$e^{2|V_{j_a}|} = \sum_{k=0}^{p_a} \frac{(2|V_{j_a}|)^k}{k!} + \int_0^1 dt_{j_a} (1-t_{j_a})^{p_a} \frac{(2|V_{j_a}|)^{p_a+1}}{p_a!} e^{2t_{j_a}|V_{j_a}|}.$$

2. Crude non-perturbative bound: (Quadratic bound)

$$\int \prod_{a\in\mathcal{B}} e^{2|V_{j_a}(\vec{\sigma}^a)|}\, d\nu_{\mathcal{B}} \leqslant K^{|\mathcal{B}|} e^{K'\rho M^{j_1}}.$$

3. Power counting (via quartic bound) beats both combinatorics and the crude non-perturbative bound. □

Perturbative bound

Theorem

Let $\mathcal{B}$ be a Bosonic block. Then there exists $K \in \mathbb{R}_+^*$ such that

$$I_{\mathcal{B}}^P(\mathrm{G}) = \int |A_{\mathrm{G}}(\vec{\sigma})|^2 \, d\nu_{\mathcal{B}} \leqslant K^{|\mathcal{B}|-1} \big((|\mathcal{B}|-1)!\big)^{37/2} \rho^{e(\mathrm{G})} \prod_{a \in \mathcal{B}} M^{-\frac{1}{48} j_a}.$$

- $A_{\mathrm{G}}(\vec{\sigma})$ depends on σ (essentially) through resolvents.
- If not for resolvents, A_{G} would be the amplitude of a convergent graph.
- Remove resolvents with iterated Cauchy-Schwarz estimates.