Multi-Time Formalism in Quantum Field Theory

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Outline

Multi-Time wave functions

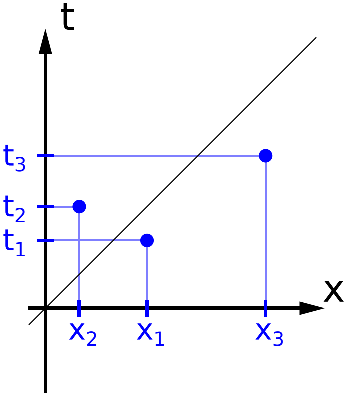
- state in Schrödinger picture:

$$|\Psi_t\rangle = \Psi(t, \boldsymbol{x}_1, ..., \boldsymbol{x}_N)$$

- perform Lorentz boost
- $\Psi'(t, \boldsymbol{x}'_1, ..., \boldsymbol{x}'_N)$ is unclear!
- introduce separate time for each particle:

$$\phi(q) = \phi(t_1, \boldsymbol{x}_1, ..., t_N, \boldsymbol{x}_N)$$

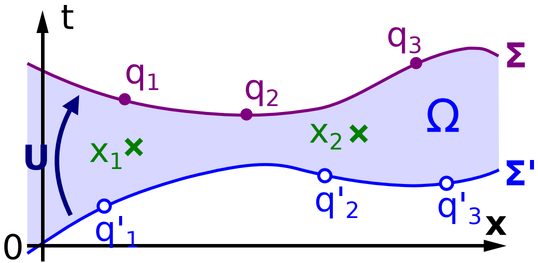
- **"Multi-Time wave function"** (Dirac, 1932)



Multi-Time wave functions

- $\phi(q) = \phi(x_1, ..., x_N) = \phi(t_1, \boldsymbol{x}_1, ..., t_N, \boldsymbol{x}_N)$
- recovery of single-time wave function:
$$\Psi_t(\boldsymbol{x}_1, ..., \boldsymbol{x}_N) = \phi(t, \boldsymbol{x}_1, ..., t, \boldsymbol{x}_N)$$
- usually only defined for space-like separated particles:
$$\|\boldsymbol{x}_j - \boldsymbol{x}_{j'}\| > |t_j - t_{j'}| \quad \Leftrightarrow: \quad q \in \mathscr{S}$$
- equations of motion:
$$i\partial_t \Psi = \boldsymbol{H}\Psi \qquad \rightarrow \qquad \begin{matrix} i\partial_{t_1}\phi(q) = H_1\phi(q) \\ \dots \\ i\partial_{t_N}\phi(q) = H_N\phi(q) \end{matrix}$$
- Hamiltonian has to be split : $H = \sum_{j=1}^N H_j$
- consistency condition: $\left[H_j - i\partial_{t_j}, H_{j'} - i\partial_{t_{j'}}\right] = 0$

What time dynamics should look like:



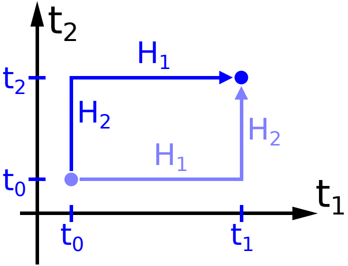
- We would like to make sense of:

$$\phi(q) = \int_{\mathcal{Q}} dq' \underbrace{\left(\sum_{n=0}^{\infty} \int_{x_k \in \Omega} d^{4n}x \, \mathcal{T}\big(H(x_1) \cdots H(x_n)\big) \right)}_{U(q,q')} \phi(q')$$

The consistency condition

- first, consider $\frac{\partial H_i}{\partial t_j} = 0$:
- unitary time evolution U depends on order of time increase

$$U_{12} = e^{-iH_2 t_2} e^{-iH_1 t_1}$$
$$U_{21} = e^{-iH_1 t_1} e^{-iH_2 t_2}$$



$$\Rightarrow \quad U_{21} - U_{12} = \left[e^{-iH_1 t_1}, e^{-iH_2 t_2} \right] \overset{!}{=} 0$$

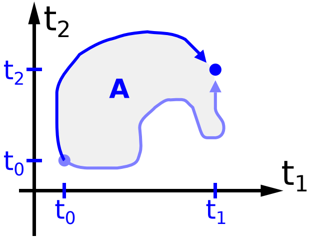
$$\boxed{[H_1, H_2] = 0}$$

Mathematical proof (bounded H_i)

- Take arbitrary paths ($\frac{\partial H_i}{\partial t_j} \neq 0$):

$$U_1 = \mathcal{T} \exp\left(-i \int_{\gamma_1} H_j(s) \cdot \dot{\gamma}_1^j(s) ds\right)$$

$$U_2 = \mathcal{T} \exp\left(-i \int_{\gamma_2} H_j(s) \cdot \dot{\gamma}_2^j(s) ds\right)$$



- and set them equal:

$$1 \stackrel{!}{=} \frac{U_1}{U_2} = \mathcal{T} \exp\left(-i \int_{\gamma} H_j(s) \cdot \dot{\gamma}^j(s) ds\right) \qquad \gamma = \gamma_1 \diamond \gamma_2^{-1}$$

$$\Leftrightarrow 1 = \mathcal{T} \exp\left(-i \int_A \left(\frac{[H_1, H_2]}{i} + \frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1}\right) dA\right)$$

- **consistency condition**: $\boxed{[H_1, H_2] + i\frac{\partial H_1}{\partial t_2} - i\frac{\partial H_2}{\partial t_1} = 0}$

Interacting potentials

- M particles with interaction potential:

$$H = \sum_{j=1}^{M} H_j^{free} + \sum_{\substack{k,j=1 \\ k \neq j}}^{M} V(x_j - x_k)$$

e.g. $$H_j^{free} \in \left\{ -\frac{\Delta_j}{2m},\ -i\alpha_a \partial_j^a + m\beta,\ \sqrt{-\Delta_j + m^2},\ |\nabla_j| \right\}$$

$$V(x_j - x_k) = \frac{1}{2\|x_j - x_k\|}$$

- splitting is simple: $H_j = H_j^{free} + \sum_{\substack{k=1 \\ k \neq j}}^{M} V(x_j - x_k)$

Interacting potentials

- Hamiltonians with interacting potentials **violate consistency**:

$$H_j = -\frac{\Delta_j}{2m} + \sum_{k \neq j} \frac{1}{2\|x_j - x_k\|}$$

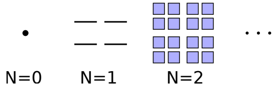
- consistency is:

$$0 \stackrel{!}{=} [H_j, H_k] = \frac{(x_j - x_k) \cdot (\nabla_j + \nabla_k)}{2m\|x_j - x_k\|^3} \neq 0 \quad \lightning$$

- Happens with **all** Lorentz-Invariant potentials! [Petrat, Tumulka (2014)], [Nickel, Deckert (2016)]

An (almost)-consistent QFT model

- M spin-$1/2$ fermions (x_k) and $N \in \mathbb{N}_0$ spin-$1/2$ bosons (y_l)
- configuration space with spin:
  $\mathcal{Q} = \left((\mathbb{R}^3)^4\right)^M \times \bigcup_{N=0}^{\infty} \left((\mathbb{R}^3)^4\right)^N$



- wave function $\Psi_t : \mathcal{Q} \to \mathbb{C}$

$$\Psi_t(\boldsymbol{q}) = \Psi^{(N)}_{r_1,\dots,r_M,s_1,\dots,s_N}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_M, \boldsymbol{y}_1, \dots, \boldsymbol{y}_N)$$

- free Dirac evolutions:

$$\boldsymbol{H}^{free}_{x_k} \Psi_{r_k} = \left(-i \sum_{a=1}^{3} (\alpha^a)_{r_k r'_k} \partial_{x_k^a} + m_x (\beta)_{r_k r'_k}\right) \Psi_{r'_k}$$
$$\boldsymbol{H}^{free}_{y_l} \Psi_{s_l} = \left(-i \sum_{a=1}^{3} (\alpha^a)_{s_l s'_l} \partial_{y_l^a} + m_y (\beta)_{s_l s'_l}\right) \Psi_{s'_l}$$

- boson annihilation by x_k: use cutoff with $supp(\varphi_\delta) \subset B_\delta(0)$

$$\left(\boldsymbol{a}_s(\boldsymbol{x}_k^{op})\Psi\right)^{(N)}(\boldsymbol{q}) = \sqrt{N+1}\int d^3\tilde{\boldsymbol{y}}\,\varphi_\delta(\tilde{\boldsymbol{y}} - \boldsymbol{x}_k)\Psi^{(N+1)}_{s_{N+1}=s}(\boldsymbol{q}, \tilde{\boldsymbol{y}})$$

- boson creation by x_k:

$$\left(\boldsymbol{a}_s^\dagger(\boldsymbol{x}_k^{op})\Psi\right)^{(N)}(\boldsymbol{q}) = \frac{1}{\sqrt{N}}\sum_{l=1}^N \delta_{ss_l}\varphi_\delta(\boldsymbol{y}_l - \boldsymbol{x}_k)\Psi^{(N-1)}_{\widehat{s_l}}(\boldsymbol{q}\backslash\boldsymbol{y}_l)$$

- interaction = creation + annihilation:

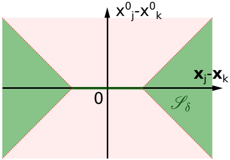
$$\boldsymbol{H}_{x_k}^{int} = \sum_{s=1}^4 (g_{s,k}\boldsymbol{a}_s(\boldsymbol{x}_k^{op}) + g_{s,k}^*\boldsymbol{a}_s^\dagger(\boldsymbol{x}_k^{op}))$$

- full Hamiltonian:

$$(\boldsymbol{H}\Psi)^{(N)} = \left(\sum_{k=1}^M \left(\boldsymbol{H}_{x_k}^{free} + \boldsymbol{H}_{x_k}^{int}\right)\Psi + \sum_{l=1}^N \boldsymbol{H}_{y_l}^{free}\Psi\right)^{(N)}$$

- Would be consistent without cutoff [Petrat, Tumulka (2014)]
- Cutoff allows for rigorous construction of a unique solution to Multi-time equations of motion [Lill (2018)]

Multi-Time



- challenge: define admissible wave functions
- only *space-like configurations* $q \in \mathscr{S}_\delta$
- particles close together are forced to equal times $t_1, ..., t_J$
- admissible wave functions:
  1. partial **derivatives** $\partial_{x_k^a}, \partial_{y_l^a}, \partial_{t_j}$ to arbitrary order are continuous
  2. define $\boldsymbol{H}_f = d\Gamma(\sqrt{\boldsymbol{k}^2 + m^2})$, $\boldsymbol{N} = \sum_k(-\boldsymbol{\Delta}_k) + \boldsymbol{H}_f + 1$
     Now, $\Psi_t \in dom(\boldsymbol{N}^n) \ \forall n \in \mathbb{N}$
     $\Rightarrow$ sector sum $\|\partial_{\boldsymbol{\alpha}} \Psi_t\|_2 = \sum_{N=0}^{\infty} \|\partial_{\boldsymbol{\alpha}} \Psi_t^{(N)}\|_{k,2} < \infty$ is finite
     $\Rightarrow$ finite **Sobolev** norms
  3. 3D-support $\mathbb{R}^3 \supset supp_3 \Psi_t$ is **compact**
- we write: $\boxed{\phi \in C^\infty_{P,c}}$ and $\boxed{\Psi_t \in \mathscr{H}^\infty_c}$

The Initial Value Problem

- IVP to be solved:

$$\phi(0, \boldsymbol{x}_1, ..., 0, \boldsymbol{y}_N) = \phi_0(\boldsymbol{x}_1, ..., \boldsymbol{y}_N) \in \mathscr{H}_c^\infty$$

$$i\partial_{t_1}\phi(q) = H_1\phi(q) = \left(\sum_{x_k \in P_1} H_{x_k} + \sum_{y_l \in P_1} H_{y_l}\right)\Psi(q)$$

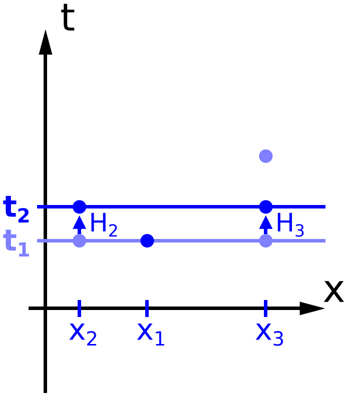
$$...$$

$$i\partial_{t_J}\phi(q) = H_J\phi(q) = \left(\sum_{x_k \in P_J} H_{x_k} + \sum_{y_l \in P_J} H_{y_l}\right)\phi(q)$$

- Theorem 1: A unique solution $\phi \in C^\infty_{P,c}$ exists $\forall q \in \mathscr{S}_\delta$

Assembling time evolutions

- solution: assemble single-time evolutions
- start with $\Psi(t_0, \boldsymbol{x}_1, ..., \boldsymbol{x}_M)$, sort $x_1^0 \leq ... \leq x_M^0$
- evolve with $H = \sum_{j=1}^M H_j$ first up to x_1^0
- evolve only with $H = \sum_{j=2}^M H_j$ up to x_2^0
- proceed with decreasing sums until x_M^0 is reached
- works if $\sum_{j=k}^M H_j,\ k \in \{1, ..., N\}$ are ess. self-adjoint



Essential Self-Adjointness

- Lemma 1: $\boldsymbol{H}$ is essentially self-adjoint on $\mathscr{H}_c^\infty$
- idea of proof: *commutator theorem* - need to find $\boldsymbol{N}$, essentially self-adjoint on $\mathscr{H}_c^\infty$

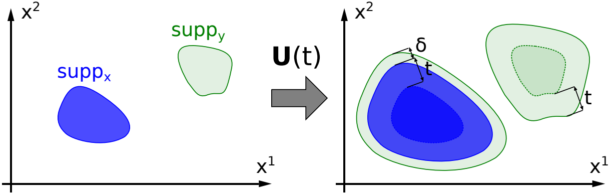
$$\|\boldsymbol{H}\Psi\| < c\|\boldsymbol{N}\Psi\|$$

$$|\langle \boldsymbol{H}\Psi, \boldsymbol{N}\Psi\rangle - \langle \boldsymbol{N}\Psi, \boldsymbol{H}\Psi\rangle| \leq d\|\boldsymbol{N}^{1/2}\Psi\|^2$$

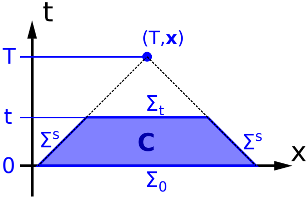
- choose $\boldsymbol{N} = \sum_{k=1}^{M}(-\boldsymbol{\Delta}_k) + \boldsymbol{H}_f + 1$, compute.
- allows use of Single-time evolutions $\boldsymbol{U}(t) = e^{-it\boldsymbol{H}}$ by Stone

Support growth

- next: show for $\Psi_0 \in \mathscr{H}_c^\infty$ that $\Psi_t = \boldsymbol{U}(t)\Psi_0 \in \mathscr{H}_c^\infty$.
- Lemma 2: 3D-supports do not grow faster than light:



- idea of proof: probability current argument + Stokes Thm.
- $\int_C \partial_\mu j^\mu = 0 \Rightarrow \int_{\partial C} \boldsymbol{n} \cdot \boldsymbol{j} = 0$
- $\boldsymbol{n} \cdot \boldsymbol{j} \geq 0$ everywhere $\Rightarrow$ $\boxed{j^\mu = 0}$



Smoothness

- Lemma 3: $\Psi_0 \in \mathscr{H}_c^\infty$ implies smoothness of Ψ_t
- idea of Proof: *theorem by Huang:*

  boundedness of $\quad \boldsymbol{Z}_{n'} = \boldsymbol{N}^{n'-1}[\boldsymbol{H}, \boldsymbol{N}]\boldsymbol{N}^{-n'}$

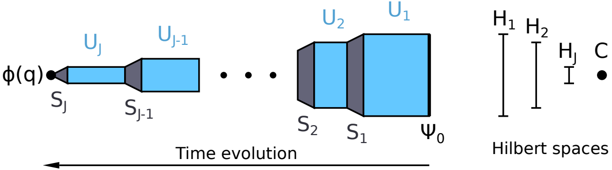
  implies $\quad \boldsymbol{U}(t)[dom(\boldsymbol{N}^n)] = dom(\boldsymbol{N}^n)$

- smoothness follows by Sobolev embedding

- Ψ stays smooth (Property 1); $\Psi_t \in dom(\boldsymbol{N}^n)$ (Property 2)
- By Lemma 2, 3D-support stays compact (Property 3)
- $\Rightarrow \Psi$ stays in $\mathscr{H}_c^\infty$

Solution construction

- combine single-time evolutions by $\boldsymbol{U}_j(t) = e^{-it\boldsymbol{H}_{j..J}}$
- each $\boldsymbol{U}_j(t) = e^{-it\boldsymbol{H}_{j..J}}$ acts on a different Hilbert space $\mathscr{H}_j$, so a formal "stacking" $\boldsymbol{S}_j : \mathscr{H}_j \to \mathscr{H}_{j+1}$ is needed

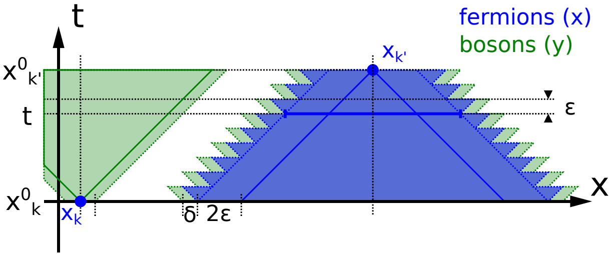


- point-wise construction for $q \in \mathscr{S}_\delta$ (fails on a null set):

$$\phi(q) := \frac{1}{\sqrt{N!}} \bigcirc_{j=1}^{J} \left(\boldsymbol{S}_j \boldsymbol{U}_j(t_j - t_{j-1})\right) \Psi_0$$

Existence

- Lemma 4: $\phi(q)$ solves the Multi-time equations (almost everywhere)
- idea of proof: direct computation:
  $i\partial_{t_j}\phi = \frac{1}{\sqrt{N}}(\boldsymbol{S}_J\boldsymbol{U}_J)...(\boldsymbol{S}_j\boldsymbol{H}_j\boldsymbol{U}_j)...(\boldsymbol{S}_1\boldsymbol{U}_1)\Psi_0 = H_j\phi$
- support cutoff prevents unwanted interactions



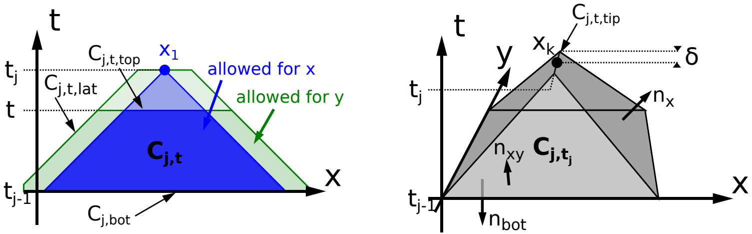
- note: proof fails on a null set!

Characteristics

- Lemma 5: $\boxed{\phi \in C^{\infty}_{P,c}}$
  1. ϕ is smooth: can be inferred from Lemma 3
  2. $\Psi_t \in dom(\boldsymbol{N}^n)$: direct consequence of Lemma 3
  3. 3D support of Ψ_t is compact: follows by Lemma 2
- Sobolev embedding $\Rightarrow$ solution $\phi(q)$ extends to the null set and Multi-Time equations are solved for all $q \in \mathscr{S}_\delta$.

Uniqueness

- Lemma 6: $\Psi_0 = 0$ implies $\phi(q) = 0 \; \forall q \in \mathscr{S}_\delta$
- idea of proof: probability argument + Stokes (again)



- now, $\partial_\mu j^\mu \neq 0$, but $\int_C \partial_\mu j^\mu = 0$
- Lemmas 4 and 6 together conclude the proof of Theorem 1.

Open questions

- existence and uniqueness of solution have been established for toy model
- missing: creation/annihilation of fermion pairs or ϕ^3, ϕ^4 interactions
- UV-cutoff φ_δ has to be removed
- spin-1/2 bosons to be replaced by spin-1 - but conserved probability current is missing. IR-problems may appear.
  $\rightarrow$ Possible solution: Kulish-Faddeev-Transformation

Further reading:

[1] P. A. M. Dirac, V. A. Fock, B. Podolsky: *On Quantum Electrodynamics.* Physikalische Zeitschrift der Sowjetunion, 2(6):468 - 479 (1932).

[2] D. A. Deckert, L. Nickel: *Rigorous formulation of a multi-time model of fermions interacting via a quantized field by Dirac, Fock, and Podolsky* (unpublished as of September 2018).

[3] S. Petrat, R. Tumulka: *Multi-Time Schrödinger Equations Cannot Contain Interaction Potentials.* Annals of Physics 345: 17–54 (2014) http://arxiv.org/abs/1308.1065

[4] M. Lienert, S. Petrat, R. Tumulka: *Multi-Time Wave Functions.* Journal of Physics: Conference Series. Vol. 880. No. 1. IOP Publishing (2017) https://arxiv.org/abs/1702.05282