Some ideas of K.-H. Rehren and their ramifications

Michael Müger
Radboud University, Nijmegen, NL

York, April 6, 2017

Organization

1. Laudatio
2. My involvement with KHR
3. More recent work / work in progress

KHR's career

- Born 1956 in Celle.
- Studies physics in Göttingen, Heidelberg und Freiburg.
- 1979 diploma at Heidelberg.

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- 1979: First paper, on sigma-models with Klaus Pohlmeyer (1938-2008, emer. 2004, PhD with "Feldverein" Lehmann)
- 1980-1984: PhD work at Freiburg w. Pohlmeyer: "Zur invarianten Quantisierung des relativistischen freien Strings"
- Four papers (appeared 1986-88) on quantization of Nambu-Goto string (of which 3 with Pohlmeyer). Returns to the subject in 2003 for one paper with Catherine Meusburger

Postdoc

- 1984-88: postdoc at Free University Berlin.
- Changes subject to (A)QFT in low dimensions.
- 1987: First joint paper with Bert Schroer (in total ≥ 6)
- Among these, two well-known FRS papers with Fredenhagen 1989, 1992: First papers rigorously establishing the role of the 'brand new' (Joyal-Street 1986) braided tensor categories in DHR style QFT.
- FRS I/II: each $\sim$50 citations on mathscinet, 404 resp. 178 on Google Scholar

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- FRS I/II: each ∼50 citations on mathscinet, 404 resp. 178 on Google Scholar
- 1988-1990: Postdoc at Utrecht University.
- 1990: In a (not well enough known) paper, anticipates Turaev's modular categories (1992-4) by proving (among other things) that a braided category without degenerate (transparent, central) objects gives rise to a (projective) repres. of SL(2,Z). Conjecture that led to my PhD subject.

Hamburg

- 1990-1997: Hochschulassistent at II. Inst. f. Theor. Phys., Univ. Hamburg.
- 1991: Habilitation at Free University Berlin.
- Winter term 1992-93: 'Professurvertretung' at Osnabrück (position left vacant by John Roberts' move to Rome)

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- 1993: assumes first PhD student…
- 1995: With R. Longo: 'Nets of subfactors' paper (120 cit. on mathscinet, 243 on Google Scholar). Extensions of QFTs, but also 'Longo-Rehren subfactor', closely related to Ocneanu's asymptotic subfactor, the Drinfeld center in category theory etc. (This is one avatar of a very basic object in fusion categ. theory.) Most of the papers having "Rehren" in the title refer to the LR subfactor.

- 1997: Moves to University Göttingen.
- 2000-2: "Rehren duality" (relation between local nets of observables and their restriction to a boundary)
- 2000: "Algebraic holography", "A proof of the AdS-CFT correspondence", "Local Quantum Observables in the Anti de Sitter-Conformal QFT Correspondence" (PRL)
  2002/3: Two papers on the subject with Michael Duetsch
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- Since 2004 with Longo, then Bischoff: Boundary CFT.
- Obviously, this was a small selection of KHR's ~ 70 publications: Bounded Bose fields, modular objects for disjoint intervalls, "Comments on a recent solution to Wightman's axioms", . . .

KHR's PhD students

- Michael Müger (Univ. Hamburg 1997)
- Sören Köster (Univ. Göttingen 2003)
- Antonia Kukhtina (née Miteva) (Göttingen 2011)
- Daniela Cadamuro (Göttingen 2012, now Munich)
- Holger Knuth (Göttingen 2012)
- Christoph Solveen (Göttingen 2012)
- Gennaro Tedesco (Göttingen 2014)
- Luca Giorgetti (Göttingen 2016, now Rome)

KHR’s PhD students

Sören Köster

Innovation Manager bei Superior Graphite

Superior Graphite • The Open University

Bonn Area, Germany • 67

Send InMail Connect

Research and Development and pilot facility managementTalent and performance developmentDevelopment and supervision of procedures and trials Capability and budget planningEHS supervision Scheduling and… See more

Innovation Manager

Superior Graphite

Oct 2010 – Present • 6 yrs 6 mos • Koblenz

High-temperature process R&D management (since September 2012)
Superior Graphite Europe process engineering (since March 2012)
Process development project management(since October 2010)
Innovation management (since October 2010)
Sales silicon carbide powders Europe (April 2011 to June 2013)

Statistik-Charaktere

Dissertationsprojekt M. Müger
Anleitung: K.-H. Rehren

1 Kurzbeschreibung des Projektes

Der Statistik-Charakter eines Superauswahl-Sektors einer lokalen Quantenfeld-Theorie ist gegeben durch die Statistik-Monodromie mit allen anderen Sektoren der Theorie [6, 3]. Im Standard-Fall mit Permutationsgruppen-Statistik (wie sie etwa in allen 4-dimensionalen Theorien auftritt) sind alle Monodromien und damit die Charaktere trivial. Dagegen weist die Matrix der Statistik-Charaktere in 2-dimensionalen konform-invarianten Modellen mit Zopfgruppen-Statistik eine sehr interessante mathematische Struktur auf, die sowohl
(a) das Verhalten der Zustandssumme unter modularen Transformationen [12, 5] der „Temperatur“ beschreibt, als auch
(b) die Fusionsregeln (Zusammensetzung von Superauswahl-Sektoren) elementar zu berechnen erlaubt [5, 4].

Die Eigenschaft (b) verallgemeinert die Charakter-Tafel einer (endlichen) Gruppe, und es liegt nahe, die Statistik-Charaktere als Signal einer den Superauswahl-Sektoren zugrundeliegenden Quanten-Eichsymmetrie (erster Art) zu deuten. Eine solche Interpretation wird gestützt durch die Beobachtung [1, II], daß man nicht-lokale Ladungsoperatoren finden kann, die die lokalen Observablen invariant lassen und deren Werte in den irreduziblen Sektoren gerade durch die Matrix der Statistik-Charaktere gegeben sind.

Die genannte Struktur dieser Matrix kann sogar ganz allgemein in nieder-dimensionalen lokalen Quantenfeld-Theorien mit lokalisierbaren Ladungen hergeleitet werden; dabei muß jedoch die Zusatzvoraussetzung gemacht werden, daß die Matrix der Statistik-Charaktere nicht entartet ist. Es erhebt sich die folgende Frage: *Was passiert im Falle einer teilweisen Entartung? Wie ist diese Situation in ihrer Mittelstellung zwischen realistischen 4-dimensionalen Teilchen-Theorien und den konform-invarianten Modellen zu verstehen?*

Algebraic quantum field theory: $O \mapsto \mathcal{A}(O)$ satisfying axioms (isotony, locality, ...).
Doplicher-Haag-Roberts ($\sim 1970, d \geq 2+1$): Symmetric tensor category (STC) $\mathrm{Rep}\, A$ of (compactly localized) representations.
(Buchholz-Fredenhagen: general. to string-like localized charges.)

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What is an STC? Think of $\mathrm{Rep}\, G$, where G is compact group.
Tensor product: $\pi, \pi' \rightsquigarrow \pi \otimes \pi'$.
Symmetry: $c_{\pi,\pi'} : \pi \otimes \pi' \stackrel{\cong}{\to} \pi' \otimes \pi$ satisf. $c_{\pi',\pi} \circ c_{\pi,\pi'} = \mathrm{id}$.

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DHR: If unbroken compact symmetry group G acts on QFT B, and B^G is the fixed point theory ('orbifold' theory) then

- $\mathrm{Rep}\, G \hookrightarrow \mathrm{Rep}\, B^G$,
- If $\mathrm{Rep}\, B$ is trivial then $\mathrm{Rep}\, B^G \simeq \mathrm{Rep}\, G$ (as STC).

DR 1980s: Proved the converse: Given QFT A ($d \geq 2+1$),

- there is compact group G s.th. $\mathrm{Rep}\, A \simeq \mathrm{Rep}\, G$ (as STCs)
- there is a QFT B with unbroken action of G s.th. $B^G = A$ and $\mathrm{Rep}\, B$ trivial (CDR 2001).

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From now: $d = 1+1$ or $d = 1$ $(S^1, \mathbb{R})$ FRS 1989: $\mathrm{Rep}\, A$ is still defined, but the symmetry equation $c_{\pi',\pi} \circ c_{\pi,\pi'} = \mathrm{id}$ cannot be proven ('lack of manouvering space'). $\leadsto$ braided tensor category (BTC).

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KHR 1990, MM 2000: For a BTC $\mathcal{C}$, define the symmetric center as full subcategory $Z_2(\mathcal{C}) = \{\pi \mid c_{\pi',\pi} \circ c_{\pi,\pi'} = \mathrm{id} \ \forall \pi'\} \subseteq \mathcal{C}$, clearly symmetric (degenerate/transparent/central objects).

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KHR 1990: If $\mathcal{C}$ is braided fusion category with $Z_2(\mathcal{C})$ trivial then the category gives rise to $\#\mathrm{Obj}(\mathcal{C})$-dimens. proj. repres. of $SL(2,\mathbb{Z})$.

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Thus my degeneracy-removing result has essentially empty domain of applicability – at the level of QFTs.

But there is a categorical version that is useful:

Thm. (MM 1998) Let $\mathcal{C}$ be a rigid braided tensor $*$-category. Then there are a rigid braided tensor $*$-category $\mathcal{D}$ with $Z_2(\mathcal{D})$ trivial and a faithful dominant braided tensor functor $\mathcal{C} \to \mathcal{D}$. (And a nice Galois correspondence.)
If $\mathcal{C}$ is finite and $\neq Z_2(\mathcal{C})$ then $\mathcal{D}$ is modular and not trivial. $\rightsquigarrow$ 'Modularization'. Idea: $\mathcal{C}/Z_2(\mathcal{C})$.

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I am now convinced that conformal field theory and the theories of subfactors and of (braided) fusion categories are thoroughly entangled and that there very few results in either of the theories that are not relevant for the others.

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- realization in CFTs?
- classify local extensions of CFTs.
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Central result (Longo-KHR 1995, Kirillov Jr.-Ostrik,...)

- Finite local extensions of a CFT A are classified by commutative algebras Γ in $\mathrm{Rep}\, A$ (more precisely Q-systems, Frobenius algebras, étale algebras).
- If $B \supset A$ corresponds to $\Gamma \in \mathrm{Rep}\, A$ then $\mathrm{Rep}\, B \simeq \Gamma - \mathrm{Mod}^0_{\mathrm{Rep}\, A}$.

Ways to obtain modular categories:

- modularization of braided fusion categs. (not symmetric!)
- quantum groups at $\sqrt{1} \leftrightarrow$ loop groups.
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  Thm.: If $\mathcal{C}$ is spherical fusion category with $\dim \mathcal{C} \neq 0$ then $Z_1(\mathcal{C})$ is modular and $\dim Z_1(\mathcal{C}) \simeq (\dim \mathcal{C})^2$.

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Coro.: Rational CFT A admits a local extension $B \supset A$ with $\mathrm{Rep}\, B$ trivial $\Leftrightarrow \mathrm{Rep}\, A \simeq Z_1(\mathcal{C})$ for some $\mathcal{C}$.

The feeling now is that the modular categories of the form $Z_1(\mathcal{C})$ are 'trivial' and should be factored out of the classification of modular categories. This is systematized by the Witt group of modular categories (DMNO). I won't go into this here (even though it is quite relevant for classification of 2d CFTs).

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Of course, this does not prevent us from studying orbifold models B^G and their representations. Orbifold inclusions $B^G \subset B$ certainly give rise to simpler structures than general inclusions $A \subset B$. Still more complications than in higher dimensions (where $\mathrm{Rep}\, B^G \simeq (\mathrm{Rep}\, B)^G$, $\mathrm{Rep}\, B \simeq \mathrm{Rep}\, B^G / \mathrm{Rep}\, G$.)

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Still true: $\mathrm{Rep}\, G \hookrightarrow \mathrm{Rep}\, B^G$.
But: $\dim \mathrm{Rep}\, B^G = |G|^2 \dim \mathrm{Rep}\, B$ (instead of $\dim \mathrm{Rep}\, B^G = |G| \dim \mathrm{Rep}\, B$).

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Holomorphic orbifolds: $\mathrm{Rep}\, B^G \simeq D^\omega(G)\text{-}\mathrm{Mod}$, $[\omega] \in H^3(G, \mathbb{T})$.
Note: $D^\omega(G)\text{-}\mathrm{Mod} \simeq Z_1(\mathcal{C}(G, [\omega]))$.

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The last example shows that $\mathrm{Rep}\, B$ and G (which acts on $\mathrm{Rep}\, B$) may not determine $\mathrm{Rep}\, B^G$!

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Evans-Gannon (2017): For every finite group G and every $[\omega] \in H^3(G, \mathbb{T})$, the modular category $D^\omega(G)\text{-Mod}$ is realized in a CFT!

Goal: Identify additional information on B allowing to compute $\mathrm{Rep}\, B^G$.
How to do this became clear after Turaev (and others independently) invented braided G-crossed categories (2000):
Defin.: A G-crossed tensor category is a

- tensor category $\mathcal{C}$,
- carrying G-action: $X \mapsto {}^g X$.
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A braiding on a G-crossed category is a family of isomorphisms $X \otimes Y \to {}^{\partial X} Y \otimes X$ s.th. ...

(In a graded tensor category, $X \otimes Y \cong Y \otimes X$ can only hold if $\partial X, \partial Y \in G$ commute!)

Thm. (MM 2004): Let B be completely rational CFT, G finite group acting on B. Then there is a braided G-crossed category $G\text{-}\mathrm{Rep}\, B$ such that

- $(G\text{-}\mathrm{Rep}\, B)_e = \mathrm{Rep}\, B$, thus modular.
- $(G\text{-}\mathrm{Rep}\, B)_g \neq \emptyset \ \forall g \in G$. (existence of 'solitons')
- $\mathrm{Rep}\, B^G \simeq (G\text{-}\mathrm{Rep}\, B)^G$.
- $G\text{-}\mathrm{Rep}\, B \simeq \mathrm{Rep}\, B^G / \mathrm{Rep}\, G$.

(In the last statement, dividing out $\mathrm{Rep}\, G$ is as in modularization, but $\mathrm{Rep}\, G \subset \mathrm{Rep}\, B^G$ is not contained in $Z_2(\mathrm{Rep}\, B^G)$ (which is trivial), which is why the l.h.s. is not braided but G-crossed braided.)

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The objects of $(G\text{-}\mathrm{Rep}\, B)_g$ are not proper (localized) representations of B, but solitons/twisted sectors that need to be taken into account to compute $\mathrm{Rep}\, B^G$.

Problem: Given a modular category $\mathcal{C}$ carring a G-action, find all braided G-crossed categories $\mathcal{D}$ with $\mathcal{D}_e = \mathcal{C}$ and $\mathcal{D}_g \neq \emptyset \ \forall g$.

- This clearly is a question of defining the right cohomological formalism.
- Existence of such a $\mathcal{D}$ for each $\mathcal{C}$ with G-action is (essentially) equivalent to an older (2003), but not very amenable, conjecture of mine.
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The related problem (with less structure) of classifying G-graded tensor categories $\mathcal{D}$ with prescribed $\mathcal{D}_e$ has been studied extensively by Etingof-Nikshych-Ostrik (2010): There is an obstruction to existence of such an extension. When the latter vanishes, the (isoclasses of) solutions form a torsor over a certain cohomology group. (Thus no distinguished solution.)

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If the conjecture is true, then for A with $\mathrm{Rep}\,A$ trivial, $\mathrm{Rep}(A^{\boxtimes N})^G \simeq D(G)\text{-}\mathrm{Mod}$. (I.e. $[\omega] = 0$.)

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My plan had been to extract information about $S_N\text{-}\mathrm{Rep}\, A^{\boxtimes N}$ from the papers by V. Kac-R. Longo-F. Xu (2004-5) on (permutation) orbifolds. (Contemporaneous with my orbifold paper, don't discuss braided G-crossed categs., but there is much overlap.) But while KLX obtain results concerning fusion rules that are consistent with what one expects, they don't proceed in quite categorical enough fashion. They certainly haven't proven that $\mathrm{Rep}\, A \simeq \mathrm{Rep}\, B$ implies $S_N\text{-}\mathrm{Rep}\, A^{\boxtimes N} \simeq S_N\text{-}\mathrm{Rep}\, B^{\boxtimes N}$ (or the corresponding result for the orbifold theories).

New approach: Begin purely categorically. I.e. for modular category $\mathcal{C}$ and $N \in \mathbb{N}$, prove that there is a braided S_N-crossed category $\mathcal{D}$ with $\mathcal{D}_e \simeq \mathcal{C}^{\boxtimes N}$ and $\mathcal{D}_g \neq \emptyset \; \forall g$.

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Note: If $\mathcal{C}$ is realized in an (operator algebraic) CFT then existence of $\mathcal{D}$ follows for all N from my results on orbifolds! Thus a counterexample to the above problem would be a counterexample to realizability! (In which I tend not to believe.)

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Only in a third step one should try to prove that $S_N\text{-}\mathrm{Rep}\, A^{\boxtimes N} \simeq \mathcal{D}(\mathrm{Rep}\, A, N)$.

Essential ingredient: Bimodule categories for tensor categories, and the tensor product of such bimodule categories (ENO 2010).
I expect that one can essentially write down what the categories $\mathcal{D}_g,\ g \neq e$ are.

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