String-Local Fields as a Solution to Inconsistent Interactions

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Overview

- Inconsistent Interactions
  - How String-Local Fields avoid Point-Local Problems
- How to Connect Point-Local and String-Local
  - Equivalence of the S-Matrix and Induced Terms
- Choice of Time-Ordered Product
  - Construction of On-Shell T-Products

Guiding Example: Massive, complex spin 3/2 field
(Rarita-Schwinger field) ψ_μ & $\overline{\psi}_\mu$

Inconsistencies: Degrees of Freedom

[Fierz, Pauli. *Proc. R. Soc. Lond. A 173* (1939)]

► Physical degrees of freedom: Spin $s \Rightarrow (2s+1)\ d.o.f.$
Spin-3/2 → 4 *physical* degrees of freedom

► From equations of motion: $(e.o.m.)^\mu =$
$[\eta^{\mu\nu}(i\partial\!\!\!/ - m) - i(\gamma^\mu\partial^\nu + \gamma^\nu\partial^\mu) + \gamma^\mu(i\partial\!\!\!/ + m)\gamma^\nu]\psi_\nu = 0$

16 components → 16 equations of first order
→ 16/2 = **8** *independent components*
(as initial conditions)

Solution: Subsidiary Conditions

[Fierz, Pauli. *Proc. R. Soc. Lond. A 173* (1939)]

- Field equations without time-derivatives
  - Constrain Cauchy-Data

- Zero component of *e.o.m.* → 2 subsidiary conditions

$$\boxed{[\eta^{0\nu}(i\partial\!\!\!/ - m) - \ldots]\psi_\nu = 0}$$

- Divergence of the *e.o.m.* → 2 subsidiary conditions

$$\gamma_\mu(e.o.m.)^\mu - 2i\partial_\mu(e.o.m.)^\mu = 0 \;\Rightarrow\; \boxed{\gamma_\mu\psi^\mu = 0}$$

- 8 – 2 – 2 = 4 *d.o.f.* ✓

Problems with Interactions

[Fierz, Pauli. *Proc. R. Soc. Lond. A 173* (1939)]

- ► Problem: Interaction (with external field U^μ)

$$\mathscr{L}_0 + \mathscr{L}_I \Rightarrow (e.o.m.)^\mu = -\delta \mathscr{L}_I / \delta \overline{\psi}_\mu$$

- ► Example: $\mathscr{L}_I = -(\overline{\psi}_\nu \gamma_\mu \psi^\nu) U^\mu = j_\mu U^\mu$

- ► Less subsidiary conditions:

$(e.o.m.)^0 = 0$ ✓

$$\gamma_\mu \psi^\mu = \ldots - 2i/3m^2 \not{U} \partial_\mu \psi^\mu$$ ✖

- ► Wrong number of *d.o.f.*!

- ► Minimal Coupling: $\partial_\mu \to \partial_\mu + iU_\mu$
  - ► Right number of d.o.f., but Velo-Zwanziger problem

Inconsistencies: Renormalizability

- Scaling Degree = Degree of divergence at $x = x'$

$\langle T_0 \phi(x)\phi(x')\rangle = \int đ^4 p \, \frac{e^{-ip(x-x')}}{p^2 - m^2 + i\epsilon} \propto p^2$ for $p \to \infty \quad \Rightarrow s.d.(\phi) = 1$
$\delta(x - x') = \int đ^4 p \; e^{-ip(x-x')} \propto p^4 \Rightarrow s.d.(\delta) = 4$
$s.d.(\partial A) = s.d.(A) + 1$

$$\mathscr{L}_I = -(\overline{\psi}_\nu \gamma_\mu \psi^\nu) U^\mu$$

$$s.d.(\psi) = 5/2 \quad \Rightarrow \quad \boxed{s.d.(\mathscr{L}_I) = s.d.(\overline{\psi}) + s.d.(\psi) = 5}$$

- Non-Renormalizable!

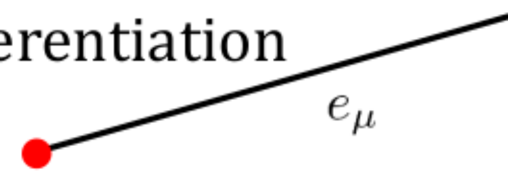
$$\hat{S} = \underbrace{\int d^4x \; T[\mathscr{L}_I(x)]}_{s.d. = 1} + \underbrace{\iint d^4x d^4x' \; T[\mathscr{L}_I(x)\mathscr{L}_I(x')]}_{s.d. = 2} + \ldots$$

String-Local Fields

[Mund, de Oliveira. *Commun. Math. Phys. 355.3* (2017)]

- The Idea: Use String-Integrals as inverse differentiation

$$I_e[f](x) = \int_0^\infty f(x+se)\ ds$$



- Define Vector Potential via $\Psi^\mu(x,e) = I_e[F^{\mu\nu}e_\nu](x)$

String-Local ➡ $\Psi^\mu(x,e) = \psi^\mu(x) + \partial^\mu I_e[e_\nu \psi^\nu](x)$ ⬅ Escort

⬆ Point-Local

- Reduces Scaling Degree to 1 (Bosons) and 3/2 (Fermions)

$$s.d. = 3/2\ \ \Psi^\mu(x,e) = \underbrace{\psi^\mu(x)}_{s.d.=5/2} + \underbrace{\partial^\mu \phi(x,e)}_{s.d.=5/2}\ \ s.d. = 3/2$$

- Axiality: $e_\mu \Psi^\mu = 0$
  (Interaction-Independent subsidiary condition)

String-Locality

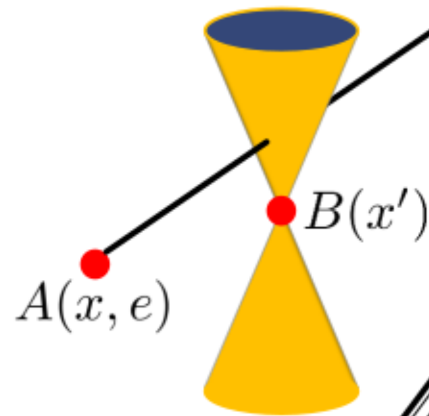
[Mund, de Oliveira. *Commun. Math. Phys. 355.3* (2017)]

- Better properties are paid for by worse locality
  - Wightman-Fields:

$$[A(x), B(x')] = 0 \ , \ \text{if } x \notin$$  $$x'$$

  - String-Local Fields:

$$[A(x,e), B(x')] = 0 \ , \ \text{if } (x+se) \notin$$  $$x' \quad \forall s \in \mathbb{R}_+$$



- Need to show: Locality!

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Connecting SL and PL Theory

[Schroer. *Nuc. Phys. B 941* (2019)]

- ►Plugging SL Fields into the PL Lagrangian: $\boxed{\psi^\mu = \Psi^\mu - \partial^\mu \phi}$

$$\mathscr{L}_I^P = -(\overline{\psi}_\nu \gamma_\mu \psi^\nu) U^\mu = -(\overline{\Psi}_\nu \gamma_\mu \Psi^\nu) U^\mu$$
$$+[(\partial_\nu \overline{\phi}) \gamma_\mu \Psi^\nu] U^\mu + \text{v.v.} - [(\partial_\nu \overline{\phi}) \gamma_\mu (\partial^\nu \phi)] U^\mu$$

- ►Partial integration & Klein-Gordon equation lower the scaling degree!

- ►L-V pair: $\underset{s.d.=5}{\mathscr{L}_I^P} = \underset{s.d.=4}{\mathscr{L}_I^s} - \underset{s.d.=5}{\partial_\nu V^\nu}$

- ►Different choices of L and V possible!
  - ► P.I. on every $\partial_\nu \phi$ (Canonical Choice)
  - ► P.I. only on the last term (Minimal Choice)

Equivalence of the S-Matrix

[Schroer. *Nuc. Phys. B 941* (2019)]

- What do we need for equivalent S-Matrices?
  - First Order:

$$S_1^P = \int d^4x \; T[\mathscr{L}_I^P] = \int d^4x \; \mathscr{L}_I^P = \int d^4x \; (\mathscr{L}_I^s - \partial V) \rightarrow \int d^4x \; \mathscr{L}_I^s = S_1^s$$

  - Second Order:

$$S_2^P = \iint T[\mathscr{L}_I^P \mathscr{L}_I^{P'}]$$

$$= \iint \left(T[\mathscr{L}_I^s \mathscr{L}_I^{s'}] - T[\mathscr{L}_I^s \partial' V'] + 2 \text{ more} \right) \stackrel{?}{=} \iint T[\mathscr{L}_I^s \mathscr{L}_I^{s'}]$$

?

- Reformulate Equivalence:

$$T[\mathscr{L}_I^P \mathscr{L}_I^{P'}] - T[\mathscr{L}_I^s \mathscr{L}_I^{s'}]$$

$$= \partial' T[\mathscr{L}_I^s V'] - T[\mathscr{L}_I^s \partial' V'] + \ldots \stackrel{?}{=} 0$$

Induced Terms

[Schroer. *Nuc. Phys. B 941* (2019)]

- More general: allow non-zero *Obstructions*
  $T[\mathscr{L}_I^P \mathscr{L}_I^{P'}] - T[\mathscr{L}_I^s \mathscr{L}_I^{s'}] = \mathcal{O}^{(2)}\ \delta(x - x')$
- String-Independent?
  → Can be re-absorbed into the Lagrangian:

$$\mathscr{L}_I^P \to \mathscr{L}_I^P - \frac{1}{2}\mathcal{O}^{(2)} \quad , \quad \mathcal{O}^{(2)} \propto g^2$$

$$T[\mathscr{L}_I^P] + \frac{1}{2}T[\mathscr{L}_I^P \mathscr{L}_I^{P'}] + O(g^3) \to T[\mathscr{L}_I^s] + \frac{1}{2}T[\mathscr{L}_I^s \mathscr{L}_I'^s] + O(g^3)$$

- String-Local reformulation changes Interaction!
  - String-Independent?
  - Higher orders?

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Freedom in Time-Ordered Products

- Obstructions $\langle T\mathscr{L}_I^s\partial' V'\rangle - \partial'\langle T\mathscr{L}_I^s V'\rangle$ depend on T-product!

- T-products must be extended to $x = x'$
- Possible extensions: $\langle T \ldots\rangle = \langle T_0 \ldots\rangle + P^{\mu\nu\cdots}(\partial)\delta(x-x')$
- Extensions...
  - ...must not exceed the scaling degree
  - ...must respect Lorentz Covariance: $P^{\cdots}(\partial) \propto \partial^\mu\ ,\ \gamma^\mu\ ,\ \eta^{\mu\nu}$

- Examples:

$$\overbrace{\langle T\partial^\mu\phi(x)\partial'^\nu\phi(x')\rangle}^{s.d.=4} = \partial^\mu\partial'^\nu\langle T_0\phi(x)\phi(x')\rangle + c\ \eta^{\mu\nu}\delta(x-x')$$

$$\overbrace{\langle T\psi_\mu\overline{\psi}'_\nu\rangle}^{s.d.=5} = \langle T_0\psi_\mu\overline{\psi}'_\nu\rangle + \Big[(c_1 + c_2\partial\!\!\!/)\eta_{\mu\nu} + \gamma_\mu(c_3 + c_4\partial\!\!\!/)\gamma_\nu + c_5\gamma_\mu\partial_\nu + c_6\gamma_\nu\partial_\mu\Big]\delta(x-x')$$

On-Shell T-Products

► Could we choose $\langle T \ldots \rangle = \langle T_0 \ldots \rangle \Rightarrow \mathcal{O}^{(2)} = 0$?

► No! $\mathscr{L}_I^P = \mathscr{L}_I^s - \partial_\nu V^\nu$ uses on-shell relation (KGE)

$$(\Box + m^2)\psi^\mu = 0 \quad \not\Rightarrow \langle T_0(\Box + m^2)\psi^\mu \, \overline{\psi}'^\nu \rangle = 0$$

► On-Shell T-products:

$$\langle T^{on} \ldots \rangle = \langle T_0 \ldots \rangle + P^{on}(\partial) \; \delta(x - x')$$

$$s.t. \; \langle T^{on}(\Box + m^2)\psi^\mu \, \overline{\psi}'^\nu \rangle = 0$$

► Canonical choice & reduced freedom

► On-Shell Obstructions:

$$\mathcal{O}^{(2)} := \langle T^{on} \mathscr{L}_I^s \partial' V' \rangle - \partial' \langle T^{on} \mathscr{L}_I^s V' \rangle$$

On-Shell T-Products & On-Shell Fields

[Brouder, Dütsch. *Math. Phys. 49.5* (2008)]

- Different ways to implement on-shell T-products
- Constraining a general Ansatz:

$$\langle T\psi_\mu \overline{\psi}'_\nu \rangle = \langle T_0 \psi_\mu \overline{\psi}'_\nu \rangle + \Big[(c_1 + c_2 \not{\partial})\eta_{\mu\nu} + 4 \text{ more} \Big] \delta(x - x')$$

$$\gamma^\mu \psi_\mu = 0 \;\&\; \overline{\psi}_\nu \gamma^\nu = 0 \;\Rightarrow \langle T^{on} \psi_\mu \overline{\psi}'_\nu \rangle = \langle T_0 \ldots \rangle + P_{\mu\nu}(\partial, c_1, c_2)\delta(x - x')$$

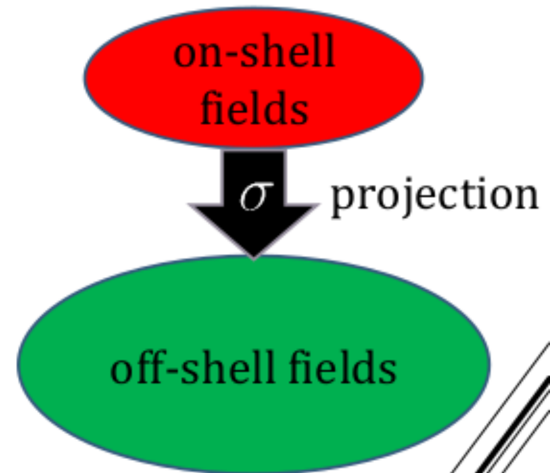
Two parameters remaining!

- On-Shell fields
  - Define on-shell fields in terms of off-shell fields:

  - Define: $\langle T^{on} \overline{\psi}_\mu \psi_\nu \rangle := \langle T_0 \sigma(\overline{\psi}^{on}_\mu) \sigma(\psi^{on}_\nu) \rangle$

BDF On-Shell Fields

[Brouder, Dütsch. *Math. Phys. 49.5* (2008)]



- Construction of on-shell fields formalized by Brouder, Dütsch and Fredenhagen
  - 1. Define on-shell relations:
    $$\mathcal{J}_1^P := \{\gamma_\mu \psi^\mu \ , \ \partial_\mu \psi^\mu \ , \ (i\partial\!\!\!/ - m)\psi^\mu\}$$
  - 2. Make ansatz for derivatives of field:
    $$\sigma(\partial^{\cdots}\psi_\mu^{on}) = \partial^{\cdots}\psi_\mu + Q_{\mu\nu}^{\cdots}(\partial)\psi^\nu$$
    $Q_{\mu\nu}(\partial)$ must only contain on-shell relations
  - 3. Fix constants:
    $$\eta^{\mu\alpha}\sigma(\partial_\alpha \psi_\mu^{on}) = 0 \ , \ i\gamma^\alpha \sigma(\partial_\alpha \psi_\mu^{on}) - m\ \sigma(\psi_\mu^{on}) = 0 \ , \ \ldots$$
- The resulting projections are *unique*!
  - More restricting than general on-shell T-products
  - 1-1 Correspondence

BDF for String-Local Fields

- For string-local fields: Derivatives & String-Integrals!

$$\langle T^{on}\overline{\Psi}_\mu(x,e)\Psi_\nu(x',e')\rangle = \langle T_0\overline{\Psi}_\mu\Psi'_\nu\rangle + P^{on}(\partial, I_e, I'_{e'})\delta(x-x')$$

- Too much freedom → BDF, but how?

- Induced construction: $\sigma(\Psi^{on}_\mu) = \sigma(\psi^{on}_\mu) + \sigma(\partial_\mu I_e e^\nu \psi^{on}_\nu)$

known!

$$\sigma(\partial_\mu I_e e^\nu \psi^{on}_\nu) = \partial_\mu I_e e^\nu \psi_\nu + Q_{\mu\rho}(\partial, I_e)\psi^\rho$$

  - String-Integrals → Infinitely many free parameters!

- Works for Spin-1! ✓
  - Unique projection, finite number of terms

- Does *not* work for Spin-3/2... ✗
  - Infinite number of undetermined constants

BDF for String-Local Fields

- Direct construction:
  - Express everything in terms of string-local fields: $\psi^\mu \to \Psi^\mu - \partial^\mu \phi$

$$\mathcal{J}_1^s := \left\{ \gamma_\mu \Psi^\mu + im\phi \ , \ \partial_\mu \Psi^\mu + m^2 \phi \ , \ (i\partial\!\!\!/ - m)\Psi^\mu \ , \ (i\partial\!\!\!/ - m)\phi \right\}$$

  - Construct projection without string-integrals:
    $\sigma(\partial^{\cdots} \Psi_\mu^{on}) = \partial^{\cdots} \Psi_\mu + Q_{\mu\nu}^{\cdots}(\partial)\Psi^\nu$
  - Additional conditions: Consistency with the pl field
    e.g. $\sigma(\partial_\alpha \Psi_\mu^{on} - \partial_\mu \Psi_\alpha^{on}) = \sigma(\partial_\alpha \psi_\mu^{on} - \partial_\mu \psi_\alpha^{on})$
- Result: Projection is *unique*!
  - String-Local: general T-products ✗ BDF ✓
  - Spin-1: Induced and direct construction coincide
  - Spin-3/2: Only direct construction works

Summary

$$\psi^\mu(x) = \Psi^\mu(x,e) - \partial^\mu \phi(x,e)$$

Renormalizable
& Consistent *d.o.f.*

$$\mathscr{L}_I^P = \mathscr{L}_I^s - \partial_\nu V^\nu$$

$$\hat{S}^P \stackrel{?}{=} \hat{S}^s$$

$$\to \mathcal{O}^{(2)} = 0$$

$$\to \mathcal{O}^{(2)} \neq \mathcal{O}^{(2)}(e)$$

$$\xrightarrow[Order]{2nd} \quad \boxed{T[\mathscr{L}_I^P \mathscr{L}_I^{P'}] - T[\mathscr{L}_I^s \mathscr{L}_I^{s'}] = \mathcal{O}^{(2)}\ \delta(x-x')}$$

$$\langle T^{on} \Psi_\mu \overline{\Psi}_\nu \rangle = \langle T_0 \sigma(\Psi_\mu^{on}) \sigma(\overline{\Psi}_\nu^{on}) \rangle$$

$$\sigma(\partial^{\cdots} \psi_\mu^{on}) = \partial^{\cdots} \psi_\mu + Q_{\mu\nu}^{\cdots}(\partial) \psi^\nu$$

On-Shell Relations

Current Results:

- BDF On-Shell T-products:
  - Spin-1 & Spin-3/2
  - Point-local & string-local ✓
- Obstructions:
  - Spin-1 (Proca-Field) ✓

    → String-Dependent (No Equivalence for chosen L-V pair)

Outlook:

- Different L-V pairs
- Obstructions for Spin-3/2

Thank you
for your Attention!

Questions?