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Mathematics in the Sciences

Supersymmetric Quantum Field Theory in Curved Space-time

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Introduction

- Supersymmetry in Flat space was developed as an extension of the Standard Model of particle physics
- Supersymmetric QFTs are mathematically better behaved
- We study rigid superconformal symmetry on a fixed curved space-time (as opposed to local supersymmetry = supergravity)
- Supersymmetry can be manifest as
  1. Symmetries of Minkowski S-matrix on Hilbert space
     [Haag-Lopuszanski-Sohnius '75,...]
  2. Algebra of transformations of an invariant Lagrangian
     e.g. $\mathcal{N} = 1, 2, 4$-gauge theories on $\mathbb{R}^{3,1}$.
  3. Conformal symmetry superalgebra on curved space-times
     [deMedeiros - Hollands '13]

Basic Idea of Supersymmetry

- Symmetry transformations of (M, g) form a Lie algebra.
- Supersymmetry transformations form a Lie superalgebra.

  (i) $\mathbb{Z}_2$-graded algebra $\mathcal{S} = \mathcal{B} \oplus \mathcal{F} =$ even $\oplus$ odd,
  (ii) Graded Lie bracket $[\mathcal{B}, \mathcal{B}] \subset \mathcal{B}, \quad [\mathcal{B}, \mathcal{F}] \subset \mathcal{F}, \quad [\mathcal{F}, \mathcal{F}] \subset \mathcal{B}$.
  (iii) Graded Jacobi identity

- For a d-dim, Lorentzian manifold (M, g), it is given by a *Conformal symmetry superalgebra* [de Medeiros - Hollands '13]

$$\mathcal{B} = \{\text{Conformal KVs } (\mathcal{L}_X g = -2\sigma_X g)\} \oplus \mathcal{R}$$

$$\mathcal{F} = \left\{\text{Twistor spinors } (\nabla_\mu \psi = \frac{1}{d}\gamma_\mu \nabla\!\!\!\!/\, \psi)\right\} \otimes W.$$

$\mathcal{R}$: real Lie algebra with constant elements (R-symmetry),
W: complex $\mathcal{R}$-module.

Classification of $\mathcal{S}$'s [deMedeiros - Hollands '13]

$(\mathscr{M}, g)$	d	type	$\mathcal{R}$	$\mathcal{S}_0(\mathbb{R}^{p,q})$	Nahm label
Lorentzian	6	$\mathbb{H}$	$\mathfrak{sp}(\mathscr{N})$	$\mathfrak{osp}(6,2\|\mathscr{N})$	X
Lorentzian	5	$\mathbb{H}$	$\mathfrak{sp}(1)$	$\mathfrak{f}(4)''$	$\mathtt{IX}_2$
Riemannian	5	$\mathbb{H}$	$\mathfrak{sp}(1)$	$\mathfrak{f}(4)'$	$\mathtt{IX}_1$
Lorentzian	4	$\mathbb{C}$	$\mathfrak{u}(\mathscr{N} \neq 4)$	$\mathfrak{su}(2,2\|\mathscr{N})$	VIII
Lorentzian	4	$\mathbb{C}$	$\mathfrak{su}(4)$	$\mathfrak{psu}(2,2\|4)$	$\mathtt{VIII}_1$
Lorentzian	3	$\mathbb{R}$	$\mathfrak{so}(\mathscr{N} \neq 1)$	$\mathfrak{osp}(\mathscr{N}\|2)$	VII
Riemannian	3	$\mathbb{H}$	$\mathfrak{u}(1)$	$\mathfrak{osp}(2\|1,1)$	$\mathtt{VII}_1$

- type: the ground field $\mathbb{K}$ over which the rep. of $\mathcal{R}$ is defined
- $\mathscr{N}$: the dimension over $\mathbb{K}$ of this representation.

Manifolds admitting twistor spinors in 4-dim. [Lewandowski '91]

1. Locally conformally flat
   (Ex. $\mathbb{R}^{1,3}, dS_4, AdS_4$)
2. pp-waves
   (can describe the region of a gravitational wave far from the source)
3. Fefferman space

$4d, \mathcal{N}=2$ Classical Field Theory

- Symmetry (rigid Superconformal $\mathcal{S}$ + local gauge symm. $C^\infty(M, \mathfrak{g})$),
- Fields $\Phi = (A_\mu, \phi, \psi)$
- Action $S = \int \frac{1}{4} F \wedge \star F + \frac{1}{2}(D\phi)^2 + \frac{1}{2}\bar{\psi} \not{D} \psi + \frac{1}{6} R \phi \phi^* + \dots$

Split $S = S_0 + \lambda S_1$. If S_0 generates hyperbolic PDEs for Φ^i

1. ▶ Deform the classical theory $(\mathbf{P}(M), \cdot) \to (\mathbf{P}[[\hbar]], \star) := W_0$

$$[\Phi(x), \Phi(y)]_\star = i\hbar\Delta(x, y).$$

   ▶ Factor out the ideal $\mathcal{J}_0$ generated by free e.o.m.

2. Perturbative interacting QFT consists of interacting fields

$$\Phi(x)_{\text{int}} := T(e_\otimes^{i\lambda S_1/\hbar})^{-1} \star T(e_\otimes^{i\lambda S_1/\hbar} \otimes \Phi(x)) \in W_0[[\lambda]].$$

- $T_n : \mathbf{P}^{\otimes n} \to W_0$: renormalization schemes (satisfy certain axioms),
- T_n exist, but is unique up to local finite counter terms $D_n = \mathcal{O}(\hbar)$.

Difficulties in our case

- Due to local gauge symm., S_0 does not generate a hyperbolic PDE!
- Solution: the (extended) BRST formalism [Becchi-Rouet-Stora, '74, Tyutin'75].

BRST Formalism

1. Enlarge the space of fields (A_μ, ϕ, ψ) to include
   - Dynamical ghosts: c, $\bar{c}$
   - Rigid ghosts: ϵ(SUSY), α (R-symmetery), X (conformal)
   - Anti-fields: $\Phi^\ddagger(x)$ associated to all fields and ghosts

   (In a GNS rep. rigid ghosts, anti fields are represented by 0 element)
2. Extend $S \to S^{\text{ext}}$ s.t. S_0^{ext} does generate a hyperbolic PDE:

$$S^{\text{ext}} = S + Y(\Phi^\ddagger)^2 + s\mathcal{G} + \int s\Phi \cdot \Phi^\ddagger$$

3. Replace the whole superconformal + (fixed) gauge symmetries with one symmetry $\hat{s} = s + \delta$ (BRST + Koszul-Tate)

$$\hat{s}S^{\text{ext}} = 0, \qquad \hat{s}^2 = 0.$$

(e.g. $\hat{s}\phi = [\phi, c] + 2\alpha\phi + (\mathcal{L}_X - \sigma_X)\phi + \bar{\epsilon}\psi$)

Quantum BRST charge

We proceed by quantizing the enlarged (non-physical) theory ...

Q : How to recover the original, physical interacting QFT?

A : the cohomology of a realization of BRST diff. on $W_0[[\lambda]]$

At quantum level, symmetries are generated by Noether charge. Therefore, if the following two criteria hold,

1. The Noether current associated to BRST symm. is conserved:

$$d\mathbf{J}_{\mathsf{int}} = 0 \Rightarrow Q_{\mathsf{int}} = \int_\Sigma \mathbf{J}_{\mathsf{int}}$$

2. $[Q_{\mathsf{int}}, -]_\star = \hat{s} + \mathcal{O}(\hbar)$ is nilpotent:

$$[Q_{\mathsf{int}}, [Q_{\mathsf{int}}, F_{\mathsf{int}}]_\star]_\star = 0 \quad \forall F_{\mathsf{int}} \in W_0[[\lambda]],$$

then, we can define the algebra of interacting fields as

$$\{\textbf{Physical Quantum Observables}\} = \frac{\mathrm{Ker}[Q_{\mathsf{int}}, -]_\star}{\mathrm{Im}[Q_{\mathsf{int}}, -]_\star}$$

Ward Identity

$d\mathbf{J}_{\text{int}} = 0$, follows from a renormalization condition (Ward Identity):

$$\boxed{\hat{s}_0 T(e_{\otimes}^{iF/\hbar}) = \frac{i}{2\hbar} T((S_0 + F, S_0 + F) \otimes e_{\otimes}^{iF/\hbar})}$$

- $F = \int f\mathcal{O}$, with $f \in C_0^\infty(M)$ being an IR cutoff function
- (F, G) is the anti-bracket, with $\hat{s} = (S, -)$.

The Ward identity ensures that

1. Generating functional of T_n respects the classical symmetry $\hat{s}_0$
2. FORMALLY, if $f \to 1$ (adiabatic limit), then

$$\int f\mathcal{L}_1 \to S_1 \text{ and } (S_0 + F, S_0 + F) \to (S, S) = \hat{s}^2 = 0,$$

$$\hat{s}_0 T(e_{\otimes}^{iS_1/\hbar}) = 0 \quad \text{“ S-matrix is BRST invariant ”.}$$

Anomaly

To prove the Ward identity,

1. set up an Anomalous WI [Hollands '07, Brennecke, Duetsch '07].

$$\hat{s}_0 T(e_\otimes^{iF/\hbar}) = \frac{i}{2\hbar} T((S_0+F, S_0+F)\otimes e_\otimes^{iF/\hbar}) + \frac{i}{\hbar} T(A(e_\otimes^F)\otimes e_\otimes^{iF/\hbar}),$$

2. try to remove the **Anomaly** $A(e_\otimes^F) = \sum_n \frac{1}{n!} A_n(F^{\otimes n})$.

$$A_n : \mathbf{P}(M)^{\otimes n} \to \mathbf{P}(M)[[\hbar]]$$

(a) each A_n is a local functional supported on total diagonal,

(b) $A_n(F^{\otimes n}) = \mathcal{O}(\hbar)$

Anomaly = failure of a classical symmetry to be a symm. of QFT

Removal of anomaly

- The lowest order expansion in $\hbar$ of anomaly A^m satisfies

$$\hat{s}A^m = 0 \Longrightarrow A^m \in H_1^4(\hat{s}|d, M)$$

- If A is the trivial element in $H_1^4(\hat{s}|d, M)$, then $A^m = \hat{s}B$.
  We can prescribe another scheme via $D = -B$.
  Then, the anomaly $\hat{A}$ in $\hat{T}(\mathrm{e}_\otimes^L) = T(e_\otimes^{L+D})$ vanishes.
- Iterate the argument for higher orders of $\hbar$.

In the case of $4d, \mathcal{N} = 2$ superconformal theory, with

$$\mathcal{S} = (\{X|\mathcal{L}_X g = 2\sigma_X g\} \oplus \mathfrak{u}(2)) \oplus \left(\{\epsilon|\nabla_\mu \epsilon = \frac{1}{4}\gamma_\mu \not{\nabla} \epsilon\} \otimes \mathbb{C}^2\right)$$

$A(e_\otimes^{S_1}) = 0$ is trivial if and only if EITHER [deMedeiros - Hollands '13]

1. Twistors are *parallel* ($\not{\nabla}\epsilon = 0$) and CKV are *Killing* ($\sigma_X = 0$) (e.g. $\mathbb{R}^{3,1}$, pp-wave, but not e.g. dS_4, Fefferman!) OR
2. $\mathcal{S}$ contains *non-parallel* twistor spinors, but $\beta = 0$.

Conservation of interacting current

In case one of the above criteria is satisfied ($A(e_\otimes^{S_1}) = 0$), then:

Theorem 1

The interacting Nother current associated to the BRST symmetry is conserved: $d\mathbf{J}_{\text{int}} = 0$.

- Therefore, there exists a well-defined (independent of the Cauchy surface Σ) interacting BRST charge $Q_{\text{int}} = \int_\Sigma \mathbf{J}_{\text{int}}$.

Theorem 2

The commutator of Q_{int} and any interacting observable F_{int} can be written as

$$\boxed{[Q_{\text{int}}, F_{\text{int}}]_\star = (\hat{s}F + A(e_\otimes^{S_1} \otimes F))_{\text{int}} \quad \text{mod } \mathcal{J}_0}$$

- Although $A(e_\otimes^{S_1}) = 0$, but $A(e_\otimes^{S_1} \otimes F) = \frac{d}{d\tau} A(e_\otimes^{S_1 + \tau F})|_{\tau=0} \neq 0$.
- Similar to an expression for local symm. in BV-formalism [Rejzner '13]

Nilpotency of BRST diff. generated by Q_{int}

When $A(e_{\otimes}^{S_1}) = 0$, we have the following corollaries:

Corollary 1

Consistency condition implies that $[Q_{\text{int}}, -]_\star$ is nilpotent:

$$[Q_{\text{int}}, [Q_{\text{int}}, F_{\text{int}}]_\star]_\star = 0, \qquad \forall F_{\text{int}} \in W_0[[\lambda]]$$

Corollary 2

The Jacobi identity implies that

$$Q_{\text{int}}^2 = \frac{1}{2}[Q_{\text{int}}, Q_{\text{int}}]_\star = 0$$

Corollary 3

Given a gauge invariant classical observable Ψ with zero ghost number, $\hat{s}\Psi = 0$,

$$[Q_{\text{int}}, \Psi_{\text{int}}]_\star = 0.$$

Outlook

- Under certain criteria, $\mathcal{N} = 2$ superconformal Yang-Mills theory in 4 dimensions on fixed Lorentzian space-times admitting twistor spinors can be consistently formulated at quantum level.
- What about dimensions $3, 5, 6$?
- Conformal symmetry superalgebra has constant R-symmetry.
- What about gauging the R-symmetry? What are the classifications of $\mathcal{S}$ with local R-symmetery? Does QFT respect local R-symmetry?
- $>$ Non-perturbative effects?