Quantum stress-energy tensors without action functional

Karl-Henning Rehren

Universität Göttingen, Institut für Theoretische Physik

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Abstract

The **stress-energy tensor** describes (among other) the coupling of matter to gravity.

Classical stress-energy tensors are usually derived by **variations of the action**. While already for Maxwell, the canonical formula gives an asymmetric and non-gauge invariant result, the Hilbert prescription (by variation of the metric) gives the correct symmetric and gauge-invariant stress-energy tensor.

For higher spins, the problems become more severe because the action has to take care of manifold constraints. In the quantum case, additional problems of indefinite metric arise on top, which lead to famous no-go results.

I present an **alternative approach** that allows to construct higher-spin stress-energy tensors **intrinsically via the Wigner representation**, without reference to an action functional. The method also applies to infinite-spin representations.

The action principle

The action principle is an extremely successful paradigm in classical mechanics and classical field theory.

Invariants $\rightarrow$ *field equations*

It became most influential for modern QFT (eg, via the path integral), and is often regarded as fundamental.

Yet, . . .

I want to shed some vinegar over this beautiful picture

Higher spin: the trouble starts

Covariant fields of spin $s \geq 1$ have more components than physical degrees of freedom. Need for kinematical and dynamical **constraints**. ($\rightarrow$ Gauge symmetry and Noether's Second Theorem)

Massive, spin one:

$$\partial_\mu A^\mu \stackrel{!}{=} 0.$$

Massive, spin two:

$$\partial_\mu A^{\mu\nu} \stackrel{!}{=} 0, \quad \eta_{\mu\nu} A^{\mu\nu} \stackrel{!}{=} 0.$$

Massless, spin one:

$$\partial_\mu A^\mu \stackrel{!}{=} 0 \quad + \text{gauge invariance.}$$

. . .

Fronsdal 1978 (**Fierz-Pauli** 1939):

Spin ≥ 2 requires at least $s-1$ **auxiliary fields**. The case $s=2$ is possible with $A=\eta_{\mu\nu}A^{\mu\nu}$ as unique auxiliary field:

$$L=\frac{1}{4}F_{[\mu\nu]\kappa}F^{[\mu\nu]\kappa}-\frac{1}{2}(\partial A)_\kappa(\partial A)^\kappa-\frac{1}{4}\partial_\kappa A\partial^\kappa A-\frac{m^2}{2}(A_{\nu\kappa}A^{\nu\kappa}-A^2)$$

(where $F_{[\mu\nu]\kappa}=\partial_\mu A_{\nu\kappa}-\partial_\nu A_{\mu\kappa}$, $(\partial A)^\kappa=\partial_\mu A^{\mu\kappa}$),

but this pattern ceases at $s>2$.

Quantization: the trouble continues

- Canonical quantization fails because some components have **no canonical conjugate momentum**. Needs "gauge fixing".
- Classical Lagrangean **not unique**: adding a total divergence to $\mathcal{L}$ preserves $S = \int \mathcal{L}$, and hence the Euler-Lagrange equations of motion; but may change the path integral.
- More drastic case: Nambu-Goto string vs Polyakov string (induced metric vs worldsheet metric) yield **inequivalent quantization**.
- Still results in **indefinite** "canonical" Hilbert spaces. The latter issue has no classical analogue. Needs "ghosts" and BRST.

Even with such an arsenal of elaborate tricks: A queasy feeling remains.

Fortunately, one can construct local free fields of any mass and any finite spin **directly on the physical Fock space** (**Weinberg** 1964), based on the underlying positive-energy representations of the Poincaré group (**Wigner** 1939).

Crucial ingredient: **"intertwiners"** to mediate between the covariance of one-particle states and the covariance of fields.

Perturbative QFT along the lines of Bogoliubov and of Glaser-Epstein starts from free fields (no need of a "free Lagrangian"). The "interacting part of the Lagrangian" is just a linear space (of couplings) on which the renormalization group acts (by readjusting the coefficients).

Stress-energy tensor: more trouble

Classical Maxwell: canonical stress-energy tensor (SET) is neither symmetric nor gauge-invariant. Can be fixed "by hand".

General: **many ambiguities** (**Belinfante** 1940)

Modern attitude: **Hilbert SET** via variation of a generally covariant action by the metric. Automatically symmetric and gauge invariant.

Quantum stress-energy tensor: yet more trouble

Weinberg-Witten theorem (1980): For $s > 1$, a local covariant stress-energy tensor on the physical Hilbert space **does not exist** (even for free fields).

(Conflict between covariant transformation laws of one-particle states and the purported SET).

Then, how does massless quantum matter couple to the gravitational field?

Reminder: Wigner quantization

- $p_0 \in \mathrm{Spec}(P)$, $p = B_p p_0$.
- unirep d of "little group" $\mathrm{Stab}(p_0) \subset L_+^\uparrow$ induces one-particle unirep U_1 of $\mathrm{P}_+^\uparrow$ (Mackey).
- Second quantization $U = \Gamma(U_1)$ on Fock space $\mathcal{H} = \Gamma(\mathcal{H}_1)$.
- Creation and annihilation operators $a_m^{(*)}(p)$ on $\mathcal{H}$.
- Adjoint action $U(\Lambda)a_m^*(p)U(\Lambda)^* = a_n^*(\Lambda p)d(W_{\Lambda,p})_{nm}$, with the "Wigner rotations" $W_{\Lambda,p} = B_{\Lambda p}^{-1}\Lambda B_p \in \mathrm{Stab}(p_0)$.
- Free quantum fields

$$\phi_M(x) = \int d\mu(p)\, e^{ipx}\, u_{M,m}(p)a_m^*(p) + h.c.$$

- $\phi_M(x)$ **transform covariantly**:

$$U(\Lambda)\phi_M(x)U(\Lambda)^* = D(\Lambda^{-1})^N_M \phi_N(\Lambda x)$$

iff the intertwiner functions $u_{M,m}(p)$ satisfy

$$u(\Lambda p) = (D(\Lambda) \otimes d(W_{\Lambda,p}))u(p).$$

- $\phi_M(x)$ are **local** if $\overline{u_{M',m}(p)}u_{M,m}(p)$ are polynomial functions.

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Quantum SET: Roadmap

- Want to write

$$P_\mu = \int d^3x T_{\mu 0}(t,x)$$

$$M_{\mu\nu} = \int d^3x (x_\mu T_{\nu 0}(t,x) - x_\nu T_{\mu 0}(t,x))$$

with a symmetric and conserved tensor of local quantum fields.

- Rewrite $P_\mu = \int d\mu(p) p_\mu a_i^*(p) a_i(p)$ as

$$\iint d\mu(p_1) d\mu(p_2)\, a_m^*(p_1) \frac{(p_1+p_2)_\mu}{2} (p_1+p_2)_0 \delta(\vec{p}_1 - \vec{p}_2) \delta_{mn} a_n(p_2).$$

- Write $\delta(p_1 - p_2) = \frac{1}{2\pi} \int d^3x\, e^{-i(\vec{p}_1 - \vec{p}_2)\vec{x}}$.

- Find a decomposition of unity

$$\delta_{mn} = g^{MN} u_{Mm}(p)\overline{u_{Nn}(p)}.$$

- Rearrange

$$P_\mu = -\frac{1}{2} g^{MN} \int d^3x \Big[\int d\mu(p_1) e^{ip_1 x} u_{Mi}(p_1) a_i(p_1)^* \Big] \cdot$$

$$\overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}_0 \Big[\int d\mu(p_2) e^{-ip_2 x} \overline{u_{Nj}(p_2)} a_i(p_2) \Big].$$

- Symmetrize $1 \leftrightarrow 2 \Rightarrow$

$$P_\mu = -\frac{1}{4} g^{MN} \int d^3x \, {:}\phi_M \overset{\leftrightarrow}{\partial}_\mu \overset{\leftrightarrow}{\partial}_0 \, \phi_N{:}\,(x) \overset{!}{=} \int d^3x \; T_{\mu 0}(x).$$

- Similar (but more laborious) for Lorentz generators

$$M_{\mu\nu} = \frac{1}{2} \int d\mu(p) a_i^*(p) (p \wedge \overset{\leftrightarrow}{\partial}_p)_{\mu\nu} a_i(p)$$

$$\overset{!}{=} \int d^3x \left(x_\mu T_{\nu 0}(x) - x_\nu T_{\mu 0}(x) \right).$$

- Conclude

$$T_{\mu\nu}(x) = -\frac{1}{4} g^{MN} \, {:}\phi_M \overset{\leftrightarrow}{\partial}_\mu \overset{\leftrightarrow}{\partial}_\nu \, \phi_N{:}\,(x) + \partial_\kappa [\Delta T^\kappa_{\mu\nu}].$$

Derivative terms needed to accomodate the Lorentz generators.

Massive vs massless

The main issue in this program is to find a complete system of "localizing intertwiners" (ie. defining local quantum fields) satisfying

$$\delta_{mn} = g^{MN} u_{Mm}(p)\overline{u_{Nn}(p)}.$$

- In the massive case, this is possible for any spin, giving SETs originally found by Fierz (1939) (without the derivative terms).
- In the massless case, there is an **obstruction**.

Reminder: The Maxwell case

The origin of the obstruction is well known in the Maxwell case: A local intertwiner for a vector potential (transforming in $D(\Lambda) = \Lambda$) does not exist (**Weinberg** 1964).

The best one can achieve, is

$$U(\Lambda)A_\mu(x)U(\Lambda)^* = (\Lambda^{-1})_\mu^\nu\big[A_\nu(\Lambda x) + \partial_\nu X\big],$$

ie the vector potential transforms as a vector up to an operator-valued gauge transformation. Of course, $F_{\mu\nu}$ transforms correctly as a tensor.

The gauge-invariant Maxwell SET is quadratic in $F_{\mu\nu}$.

Higher helicity

The same obstruction (no local potentials, only local field strengths) occurs with higher helicity h, except that the local and covariant field strengths F involve **more derivatives**, so that quadratic expressions in F are incompatible with the scaling dimension of the SET.

To build a SET, one has to work with (first derivatives of) potentials, that are **either not exactly covariant or not exactly local**.

String-localized fields

Mund-Schroer-Yngvason (2004): Integrating the Maxwell field strength along a "string" extending from x to ∞, yields a **"string-localized potential"**

$$A_\mu(e,x) := \int_0^\infty ds\, F_{\mu\nu}(x+se)e^\nu.$$

Indeed, one has $\partial_\mu A_\nu(e,x) - \partial_\nu A_\mu(e,x) = F_{\mu\nu}(x)$, and the variation of $A(e,x)$ wrt the direction e is a gradient.

For fixed e, this is just the "axial gauge" choice of potential. But $A_\mu(e,x)$ coexist for all e on the Wigner Hilbert space.

Immediate consequences:

Covariance

$$U(\Lambda)A_\mu(e,x)U(\Lambda)^* = (\Lambda^{-1})_\mu^\nu A_\nu(\Lambda e, \Lambda x),$$

and **local commutativity** whenever two strings $x_1 + \mathbb{R}_+ e_1$, $x_2 + \mathbb{R}_+ e_2$ are spacelike separated.

The same generalizes (with h-fold integrations) to arbitrary helicity. The corresponding intertwiners are no longer polynomial, but involve distributional inverse powers $1/(ep)_+^h$ due to the string integrations.

Stress-energy tensors

With these intertwiners at hand, one can construct string-localized stress-energy tensors for **massless fields of any helicity**, defined on the Hilbert space of the field strength tensor.

The **Weinberg-Witten** theorem (assuming strict locality) is evaded.

J. Mund, KHR, B. Schroer: Nucl. Phys. B924 (2017) 699, Phys. Lett. B773 (2017) 625.

Thus, the coupling of higher-spin quantum matter to gravitational fields (classical or quantum) can be formulated. The perturbative theory of this coupling remains to be elaborated.

Optimist's view (Schroer): Notorious causality problems with such couplings (**Velo-Zwanziger** 1969) might be soothed.

Infinite spin

The massless “infinite spin” representations of the Poincaré group are special. In this case, even local “field strengths” do not exist (Yngvason 1970), from which “potentials” could be obtained by string integration.

Instead, **Mund-Schroer-Yngvason** (2004) have shown that **intrinsically string-localized fields** do exist. Their intertwiners $u(e, p)$ have to satisfy certain bounds in the complex spacelike hyperboloid $e^2 = -1$ to ensure local commutativity whenever two strings are spacelike separated.

M-S-Y gave integral representations for string-localized intertwiners $u(e, p)$.

Reminder: Infinite spin

The little group Stab(p_0) for massless representations ($p_0^2 = 0$) is isomorphic to $E(2) = SO(2) \ltimes \mathbb{R}^2$. If its "translations" $\mathbb{R}^2$ are non-trivially represented, then the unitary repn spaces are $L^2(\kappa S^1)$, where the Pauli-Lubanski parameter κ labels inequivalent representations.

Accordingly, the multiplicity spaces of one-particle states of a fixed momentum are $L^2(\kappa S^1)$, and wave fns and intertwiners are functions of $\vec{k} \in \kappa S^1$ (**"continuous spin"**) with discrete Fourier transform via $k_1 + ik_2 = \kappa e^{im\varphi}$ ($m \in \mathbb{Z}$, **"infinite spin"**).

Infinite Spin intertwiners

Köhler (2015) gave explicit expressions for string-localized intertwiners:

$$u(e,p)(\vec{k}) = \underbrace{e^{-i\frac{\kappa}{(ep)_+}}}_{\text{“Köhler factor”}} \cdot e^{-i\frac{(eB_pE(\vec{k}))}{(ep)_+}}.$$

Both factors are **highly singular** at $(ep) = 0$; but the product is a bounded function, ie, the Köhler factor absorbs the essential singularity of the $\vec{k}$-dependent factor, and provides the bound in the complex tube that ensures locality.

Comment

Schuster-Toro (2013) (in a different setting where the intertwiners are regarded as one-particle wave functions, and the parameter e is not given a geometric meaning) found the same expression –

without the Köhler factor and without worrying too much about the essential singularity. Locality is not addressed in their one-particle setting.

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Theorem:

It is possible to find an **infinite system** of string-localized intertwiners $u^{(r)}_{\mu_1\dots\mu_r}(e,p)(\vec{k})$ that fulfill the completeness relation

$$\sum_{r=0}^{\infty}(-1)^r u^{(r)}_{\mu_1\dots\mu_r}(e,p)(\vec{k})\overline{u^{(r)\mu_1\dots\mu_r}(e,p)(\vec{k'})} = \delta_{\kappa S^1}(\vec{k}-\vec{k'}).$$

Thus, the corresponding SET is an **infinite sum of Wick squares** $T^{(r)}$ of (derivatives of) tensor fields $\Phi^{(r)}_{\mu_1\dots\mu_r}(e,x)$.

Each $T^{(r)}$ is a string-localized Wightman field, while their sum is only a quadratic form (finite matrix elements, but divergent correlation functions).

Summary

- Intrinsic construction of quantum stress-energy tensors on the Fock space over Wigner's one-particle representations.
- Way around Weinberg-Witten theorem (no SET for massless fields of higher spin)
- First construction of a SET for infinite-spin matter.