The problem of equivalence of different gauges in external current QED

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Motivation and Strategy

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- In classical ED: change of gauge has no influence on the experimental results.
  In QED this issue is more controversial.

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Strategy:

i) Physics Part:
   1. Gauge Freedom
   2. Canonical Quantization
   3. Maxwell Fields in different Gauges

ii) Math Part:
   1. Equivalence of Observables in different gauges

Table of contents

Singular Systems

- Configuration space $\mathcal{M}$, Lagrange function $L : T\mathcal{M} \to \mathbb{C}$
- Legendre trafo $\Rightarrow$ Hamiltonian: $H(q,p) = \sum_i v^i p_i - L(q,p)$

$$\rho_L : T\mathcal{M} \to T^*\mathcal{M}, \qquad (q^i, v^i) \mapsto (q^i, p_i := \frac{\partial L}{\partial v^i})$$

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Definition

Lagrangian L is called singular if ρ_L is *not* a local isomorphism:

$$\det(\frac{\partial^2 L}{\partial v^i \partial v^j}) = 0$$

- **Problem**: Hamiltonian depends linearly on some v^a:

$$H(q^i, p_i, v^a) = \tilde{H}(q^i, p_i) - v^a \phi_a(q^i, p_i)$$

Singular Systems

- e.o.m. $\Rightarrow \{H, v^a\} = 0 \Rightarrow \phi_a \overset{!}{=} 0$
- With $M_{ab} := \{\phi_a, \phi_b\}$:

$$\phi_b \overset{!}{=} 0 \Rightarrow \frac{d}{dt}\phi_b = \{\phi_b, \tilde{H}\} + v^a M_{ab} \overset{!}{=} 0 \tag{1}$$

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- Two cases:
  1. $\{\phi_b, \tilde{H}\} \neq 0, det(M) \neq 0 \Rightarrow$ all v^a are fixed by (1)
  2. $\{\phi_b, \tilde{H}\} = 0$, some v^a are fixed by (1) depending on $rk(M)$

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Definition

A constraint ϕ_α is called *first* class if $\{\phi_\alpha, \phi_i\} = 0$ for every constraint function ϕ_i, otherwise *second* class.

Dirac Bracket

Case 1:

- only 2^{nd} class constraints
- v^a fixed: $v^a = -(M^{-1})^{ab}\{\phi_b, \tilde{H}\} \Rightarrow \frac{d}{dt}F = \{F, \tilde{H}\}_D$

Definition

Let $F, G \in C^\infty(\mathcal{M})$. Their *Dirac bracket* is:

$$\{F, G\}_D := \{F, G\} - \{F, \phi_a\}(M^{-1})^{ab}\{\phi_b, G\}_D$$

- $\mathcal{M}_{phys} \subset \mathcal{M}$ with $\{\cdot, \cdot\}|_{\mathcal{M}_{phys}} = \{\cdot, \cdot\}_D$

Gauge Freedom

Case 2:

- 1^{st} class constraints ϕ_α generate gauge transformations:

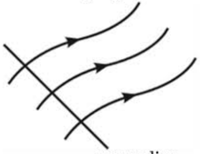
$$\delta_\epsilon F = \epsilon^\alpha \{F, \phi_\alpha\}$$

- $\{\text{gauge orbits}\} \cong \mathcal{M}_{phys}$
- Gauge fixing= intersecting each gauge orbit once

$\Leftrightarrow$ external constraints $\rightarrow$ no 1^{st} class cons.

$\Rightarrow$ Dirac bracket

[1]



[1] Graphic from H.Itoyama, *The Birth of String Theory*, Progress in Experimental and Theoretical Physics, 2016

Canonical Quantization

- Fix a Hilbert space $\mathbb{H}$
- Any $F \in C^\infty(\mathcal{M}_{phys})$ mapped to a self adjoint operator $\hat{F}$ on $\mathbb{H}$ such that:

$$\{F, G\} \to \frac{1}{i\hbar}[\hat{F}, \hat{G}]$$

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Problem: $\{\cdot,\cdot\}$ not compatible with constraints
$\Rightarrow$ Solution :

$$\{F, G\}_D \rightarrow \frac{1}{i\hbar}[\hat{F}, \hat{G}]$$

Strategy of Canonical Quantization

- $\mathfrak{h}$: one-particle space, Bosonic Fock space:

$$\Gamma_s(\mathfrak{h}) = \bigoplus_{n\geq 0} E_s(\mathfrak{h}^{\otimes n})$$

- $a(f), a^\dagger(f), f \in \mathfrak{h}$: the usual annihilation and creation operators on Γ_s

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- $a(f), a^\dagger(f), f \in \mathfrak{h}$: the usual annihilation and creation operators on Γ_s
- Choose $\mathbb{H} = \Gamma_s(\mathfrak{h})$ with $\mathfrak{h} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$
- Find a classical representation of Dirac bracket in terms of modes $\tilde{a}_n, \tilde{a}_n^\dagger$ satisfying $\{\tilde{a}_n(k), \tilde{a}_m^\dagger(k')\}_D = -i\delta_{nm}\delta^{(3)}(k-k')$
- Quantization: $\tilde{a}_n^{(\dagger)} \to a_n^{(\dagger)}$

Covariant Formulation of Maxwell Equations

- Vector field $A \in \Omega^1(Mink_4)$ and field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- Current $j \in \mathcal{S}(\mathbb{R}^3) \otimes \mathcal{C}^4$ with charge $Q = \int_{\mathbb{R}^3} d^3x\, j_0(x) = \widehat{j_0}(0)$
- Lagrange density:

$$\mathcal{L} = F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu$$
$$\Rightarrow \pi_\mu := \frac{\delta \mathcal{L}}{\delta \partial^0 A^\mu} = F_{\mu 0} = E_\mu$$

$\Rightarrow \pi_0 = F_{00} = 0 \rightarrow \mathcal{L}$ is singular

Gauge Freedom

- Two first class constraints

$$\pi_0 = F_{00} \approx 0 \qquad \nabla \cdot \pi + j_0 \approx 0 \quad \text{(Gauss law)}$$

$\Rightarrow$ two generators of gauge transformations:

$$A_0 \to A_0 + \xi$$
$$A_i \to A_i + \partial_i \chi$$

$\Rightarrow$ Well knwon $U(1)$ gauge freedom
$\Rightarrow$ 2 gauge conditions needed

The Coulomb gauge

- Choose gauge conditions:

$$\text{(i)} \quad \nabla \cdot A \approx 0, \qquad \text{(ii)} \quad \Delta A_0 + j_0 \approx 0$$

$\Rightarrow$ Dirac bracket:

$$\{A_i(x), \pi_j(y)\}_D = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right) \delta^{(3)}(x-y)$$

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Let $f \in \mathcal{S}(\mathbb{R}^3) \otimes \mathbb{R}^3$, then:

- $\pi^C(f) = a((\frac{\omega}{2})^{\frac{1}{2}} P_T(\hat{f})) + a^\dagger((\frac{\omega}{2})^{\frac{1}{2}} P_T(\hat{f})) + \langle k \cdot \hat{f}, \widehat{j_0} \rangle$
- $B^C(f) = a((2\omega)^{-\frac{1}{2}} \widehat{curl(f)}) + a^\dagger((2\omega)^{\frac{1}{2}} \widehat{curl(f)})$

with P_T: projection to $\mathfrak{h}_T := \{g \in \mathfrak{h}; k \cdot \hat{g} = 0\}$

The Axial gauge

- Choose gauge conditions

$$\text{(i)} \quad e \cdot A \approx 0 \qquad \text{(ii)} \quad e \cdot (\pi - \nabla A_0) \approx 0$$

$\Rightarrow$ Dirac bracket:

$$\{A_i(x), \pi_j(y)\}_D = \left(\delta_{ij} - \frac{e_j \partial_i}{e \cdot \nabla} \right) \delta^{(3)}(x - y)$$

Problem: $\frac{e_i \partial_j}{e \cdot \nabla} : \mathcal{S}(\mathbb{R}^3) \not\to L^2(\mathbb{R}^3)$

Smearing of the Axial gauge

- Extend phase space to have n copies of A $\rightarrow$ extends Gauge freedom to $U(1) \times \cdots \times U(1)$
- Axial gauge fixing for each A_i with gauge vector $e_i \in \mathbb{R}^3$
- Dirac bracket:

$$\{A_i(x), \pi_j(y)\}_D = \left[\delta_{ij} - \frac{1}{n}\left(\sum_{i=1}^{n} \frac{e_{i,j}}{e \cdot \nabla}\right)\partial_i\right]\delta^{(3)}(x-y)$$

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- Interpretation as Riemann sum:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{e_{k,j}}{e \cdot \nabla} = PV - \int_{S^2} d\Omega(e)\ \frac{e_j}{e \cdot \nabla} g(e)$$

for $g \in C^1(S^2)$ and $\int_{S^2} d\Omega(e)\ g(e) = 1$

Observables

Let $f \in \mathcal{S}(\mathbb{R}^3) \otimes \mathbb{R}^3$, then:

- $\pi^{ax}(f) = a(i\omega^{\frac{1}{2}} P_T \hat{\bar{f}}) + a^\dagger(i\omega^{\frac{1}{2}} P_T \hat{f}) + \langle \hat{f}, \int_{S^2} \frac{e}{e \cdot k} g(e) \widehat{j_0} \rangle$
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Note:

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3. Difference of π^C and π^{ax} only in transversal part:

$$\pi^{ax}(f) = \pi^C(f) + \langle P_T(\hat{f}), \int_{S^2} \frac{P_T(e)}{e \cdot k} g(e) \widehat{j_0} \rangle$$

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$\Rightarrow$ Inequivalence can only arise from transversal fields

Weyl operators on transversal Fock space

- Transversal Fock space: $\Gamma_T := \Gamma_s(\mathfrak{h}_T)$ with

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- $\phi(f) := \frac{1}{\sqrt{2}}\left(a(\widehat{Re(f)})) + a^\dagger(\widehat{\overline{Re(f)}})\right)$ and
$\pi(f) := \frac{1}{\sqrt{2}}\left(a(\widehat{iIm(f)})) + a^\dagger(\widehat{\overline{iIm(f)}})\right)$ are self adjoint on Γ_T^{fin}

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- The Weyl operators $e^{i\phi(f)}, e^{i\pi(g)}$ are unitary and satisfy

$$e^{i\phi(f)} e^{i\pi(g)} = e^{-\frac{i}{2}\langle f,g\rangle} e^{i(\phi(f)+\pi(g))}$$

Algebra of observables

- Weyl operators of the canonical momenta:

$$e^{i\pi_T^C(if)} = e^{i\pi(if)}$$

$$e^{i\pi_T^{ax}(if)} = e^{i\langle\int_{S^2} d\Omega(e)\frac{P_T(e)}{e\cdot k}g(e)\widehat{j_0},\hat{f}\rangle}e^{i\pi(if)}$$

- Test function space $L \subset \mathfrak{h}_T$ with P_T projection on $\mathfrak{h}_T$:

$$L := \omega^{-\frac{1}{2}} curl(\mathcal{S}(\mathbb{R}^3) \otimes \mathbb{R}^3) + i\omega^{\frac{1}{2}} P_T(\mathcal{S}(\mathbb{R}^3) \otimes \mathbb{R}^3)$$

- $\mathfrak{U} := \{W(f), f \in L\}''$ is the algebra of observables
- The set $\{W(f), f \in L\}$ is irreducible in Γ_T

Gauge Equivalence for current with vanishing charge

Theorem

$e^{i\pi^C} \cong e^{i\pi^{ax}}$ iff $\widehat{j_0}(0) = 0$

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Strategy for "$\Leftarrow$":

- $e^{i\phi(\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{P_T(e)}{e \cdot k} g(e)\widehat{j_0})} := U_g(\widehat{j_0})$ is a unitary on Γ_T iff $\widehat{j_0}(0) = 0$ due to:

$$\omega^{-\frac{1}{2}} \int_{S^2} d\Omega(e) \frac{P_T(e)}{e \cdot k} g(e)\widehat{j_0} \in \mathfrak{h}_T \Leftrightarrow \widehat{j_0}(0) = 0$$

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- $U_g(\widehat{j_0}) W(if) U_g^\dagger(\widehat{j_0}) = e^{i\langle \hat{f}, \int_{S^2} d\Omega(e) \frac{P_T(e)}{e\cdot k} g(e)\widehat{j_0}\rangle} W(if)$

$\Rightarrow$ Equivalence of the gauges if $\widehat{j_0}(0) = 0$:

$$U_g(\widehat{j_0}) e^{i\pi^C} U_g^\dagger(\widehat{j_0}) = e^{i\pi^{ax}}$$

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Strategy for "$\Rightarrow$":

Defintion

Let L' be the algebraic dual of L and $F \in L'$. An automorphism of $\mathcal{W}$ of the form

$$\gamma_F(W(g)) = e^{iF(g)} W(g)$$

is called *coherent automorphism*.

- $e^{i\pi^c}$ and $e^{i\pi^{ax}}$ are linked via the coherent automorphism with $F(f) = Im\left(\langle \int_{S^2} d\Omega(e) \frac{P_T(e)}{e \cdot k} g(e) \widehat{j_0}, \omega^{\frac{1}{2}} f \rangle\right)$

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- Equivalent Problem: $\gamma_F \cong \mathbb{I} \Rightarrow \widehat{j_0}(0)$

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Strategy:

- Construct a central sequence $W(if_\lambda) \subset \mathcal{W}$:

$$[A, \lim_{\lambda\to\infty} W(if_\lambda)] = 0$$

  for all $A \in \mathcal{W}$ such that $\|f_\lambda\| = \|f\|$

- Irreducibility: $\Rightarrow W(if_\lambda) \to c\mathbb{I}$, $c \in \mathbb{C}$
- $\|f_\lambda\| = \|f\| \Rightarrow W(if_\lambda) \to \omega_0(W(if))\mathbb{I}$ weakly
- $F(f_\lambda) \to \widehat{j_0}(0)a_f$ with $a_f \in \mathbb{R}$

$\Rightarrow \gamma_F(W(if_\lambda)) \to e^{i\widehat{j_0}(0)a_f}\omega_0(W(if))\mathbb{I}$

- We can choose f such that $a_f \neq 0$

Conclusion

- Discussed Canonical Quantization for system with Gauge Freedom
- Defined Maxwell fields on Γ_s that satisfy Coulomb gauge
- Regularized Axial gauge by Smearing
- Main Result:

$$\text{Axial gauge} \cong \text{Coulombg gauge} \Leftrightarrow \widehat{j_0}(0) = 0$$

Outlook:

- Classify Axial gauges in terms of g that are unitarily equivalent

Thank you for your attention!